PRE REGIONAL MATHEMATICAL OLYMPIAD (PRMO) - 2018

ALLEN CAREER INSTITUTE KOTA (RAJASTHAN) Path to success

Date: 19/08/2018

Max. Marks: 102 **SOLUTIONS** **Time allowed: 3 hours**

1. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n?

Ans. (17)

Sol. Let the number of pages in the first book be x

$\Rightarrow$ In second book x + 50 and in third book $\frac{3}{2}(x + 50)$

$\Rightarrow$ 1 + (x + 1) + (x + x + 50 + 1) = 1709

$\Rightarrow$ 3x + 53 = 1709

$\Rightarrow$ 3x = 1656

$\Rightarrow$ x = 552

$\Rightarrow$ Last page number = $x + x + 50 + \frac{3}{2}(x + 50)$

$= 552 + 552 + 50 + \frac{3}{2}(552 + 50)$

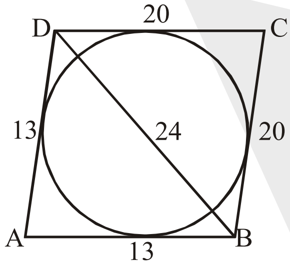
= 2057

$\Rightarrow$ Largest prime factor of 2057 = 17

2. In a quadrilateral ABCD, it is given that AB = AD = 13, BC = CD = 20, BD = 24. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

Ans. (08)

Sol.



$[ABD] = \sqrt{25(25-13)(25-13)(25-24)} = 60$,

and $[DBC] = \sqrt{32(32-20)(30-20)(32-24)} = 192$,

$\Rightarrow$ [ABCD] = 252

Now $[ABCD] = \frac{13r}{2} + \frac{13r}{2} + \frac{20r}{2} + \frac{20r}{2} = 252$

$\Rightarrow$ 33r = 252

$\Rightarrow$ $r = \dfrac{252}{33} = 7.63$

$\Rightarrow$ Nearest integer = 8

3. Consider all 6-digit numbers of the form abccba where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.

Ans. **(70)**

Sol. abccba is divisible by 7

if abc – cba is divisible by 7

$\Rightarrow$ abc – cba = 99 (a – c) = 7M $\Rightarrow$ 7|(a – c)

So, (a, c) = {(9,2), (8,1), (7,0), (2,9), (1,8), (9,9), (8,8), (7,7), (6,6), (5,5), (4,4), (3,3), (2,2), (1,1)}

No of pair of (a, b) = 14

Also, no of b's can be = 5

$\therefore$ Total number of 6 digits number = 14 × 5 = 70

4. The equation 166 × 56 = 8590 is valid in some base b ≥ 10 (that is, 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of b ≥ 10 satisfying the equation.

Ans. **(12)**

Sol. Let base be 'n'

$\Rightarrow$ $166 = 1.n^2 + 6.n^1 + 6.n^0$; $56 = 5.n^1 + 6.n^0$ and $8590 = 8n^3 + 5n^2 + 9.n^1 + 0.n^0$

Now 166 × 56 = 8590

$\Rightarrow$ $(n^2 + 6n + 6) \times (5n + 6) = 8n^3 + 5n^2 + 9n$

$\Rightarrow$ $3n^3 - 31n^2 - 57n - 36 = 0$

$\Rightarrow$ $(n - 12)(3n^2 + 5n + 3) = 0$

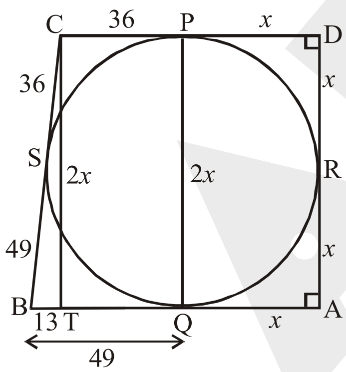
$\Rightarrow$ n = 12

So base n = 12

5. Let ABCD be a trapezium in which AB || CD and AD ⊥ AB. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49, find PQ.

Ans. **(84)**

Sol.



CP = TQ = 36 $\Rightarrow$ BT = 49 – 36 = 13; BC = BS + SC = BQ + CP = 49 + 36 = 85

In ΔBTC, $85^2 = 13^2 + (2x)^2$

$\Rightarrow$ $(2x)^2 = 7056 \Rightarrow 2x = 84$

$\Rightarrow$ PQ = 84 cm

6. Integers a, b, c satisfy a + b – c = 1 and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

Ans. **(18)**

Sol. From given equations by eliminating 'c', we get,

$a^2 + b^2 - (a + b - 1)^2 = -1$

$\Rightarrow$ $-2ab + 2(a + b) - 1 = -1$

$\Rightarrow$ $ab - a - b = 0$

$\Rightarrow$ $(a - 1)(b - 1) = 1$

$\Rightarrow$ $a - 1 = 1$ and $b - 1 = 1 \Rightarrow a = b = 2 \Rightarrow c = 3$

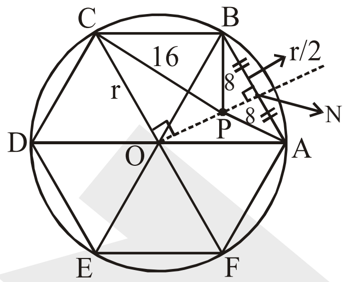
or $a - 1 = -1$ and $b - 1 = -1 \Rightarrow a = b = 0 \Rightarrow c = -1$

$\Rightarrow$ $a^2 + b^2 + c^2 = 17$ or 1

$\Rightarrow$ required sum $= 17 + 1 = 18$

7. A point P in the interior of a regular hexagon is at distance 8, 8, 16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r?

Ans. **(14)**



Sol. $ON = r\cos 30° = \frac{r\sqrt{3}}{2}$

$\Delta PCO \sim \Delta PBN$

$\Rightarrow \frac{PO}{PN} = \frac{PC}{PB} = \frac{16}{8}$

$\Rightarrow$ $PO = 2PN$

$\Rightarrow PO = \frac{2}{3}ON = \frac{2}{3}\frac{\sqrt{3}r}{2} = \frac{r}{\sqrt{3}}$

In ΔPCO, $r^2 + \frac{r^2}{3} = 16^2$

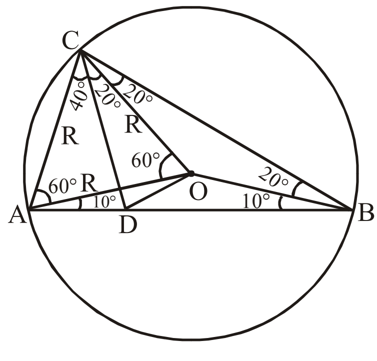
$\Rightarrow r = \sqrt{192}$

$\Rightarrow$ closest integer to r is 14.

8. Let AB be a chord of a circle with centre O. Let C be a point on the circle such that $\angle ABC = 30°$ and O lies inside the triangle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20°$. Find the measure of $\angle CDO$ in degrees.

Ans. **(80)**

Sol. $\angle ABC = 30°$



$\Rightarrow$ $AOC = 60°$

Now as $OC = OA \Rightarrow \Delta OAC$ is equilateral

$\Rightarrow$ $\angle CAO = \angle ACO = 60°$

$\Rightarrow$ $\angle ACD = 60° - 20° = 40°$

Join OB, since $OC = OB$ so

$\angle OBC = \angle OCB = 20°$

$\Rightarrow$ $\angle OBA = 10° \Rightarrow \angle OAB = 10° \Rightarrow \angle DAC = 70°$

In ΔACD by ASP

$\angle CDA = 70°$

$\Rightarrow \angle CDA = \angle CAD = 70°$

$\Rightarrow CD = CA = CO$

In ΔCDO, $CD = CO$ and $\angle DCO = 20°$,

$\Rightarrow$ $\angle CDO = \frac{180° - 20°}{2} = 80°$

9. Suppose a, b are integers and a + b is a root of $x^2 + ax + b = 0$. What is the maximum possible values of b^2?

Ans. (81)

Sol. As a + b is a root of $x^2 + ax + b = 0$, $(a + b)^2 + a(a + b) + b = 0$

$\Rightarrow\ 2a^2 + 3ba + b^2 + b = 0$

$$\Rightarrow\ a = \frac{-3b \pm \sqrt{b^2 - 8b}}{4}$$

So, $b^2 - 8b$ must be a perfect square for some whole number $= k^2$ (say), $k \in \mathbb{N}_0$

$\Rightarrow\ (b - 4)^2 - 16 = k^2$

$\Rightarrow\ (b - 4)^2 - k^2 = 16$

$\Rightarrow\ (b - 4 - k)(b - 4 + k) = 16$

Now we have following four possibilities :

(i) $b - 4 + k = 8,\ b - 4 - k = 2 \Rightarrow\ (b, k) = (9, 3)$

(ii) $b - 4 + k = 4,\ b - 4 - k = 4 \Rightarrow\ (b, k) = (8, 0)$

(iii) $b - 4 + k = -2,\ b - 4 - k = -8 \Rightarrow\ (b, k) = (-1, 3)$

(iv) $b - 4 + k = -4,\ b - 4 - k = -4 \Rightarrow\ (b, k) = (0, 0)$

Now maximum possible b = 9 and corresponding $a = \frac{-27 \pm 3}{4} = -6, -\frac{15}{2}$

As b = 9, a = –6 satisfy all constraints, maximum $b^2 = 81$.

10. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

Ans. (24)

Sol. $CE^2 = (2x)^2 + y^2$

$= 4x^2 + y^2$

and $BF^2 = (2y)^2 + x^2$

$= 4y^2 + x^2$

Also $CG^2 + BG^2 = BC^2$

$\Rightarrow\ 4x^2 + 4y^2 = 20^2$

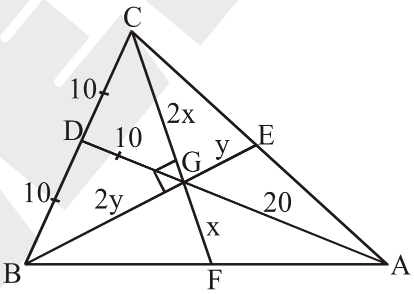
or $x^2 + y^2 = 100$

Now $AC^2 = (2CE)^2 = 4(4x^2 + y^2)$

and $AB^2 = (2BF)^2 = 4(4y^2 + x^2)$

$\Rightarrow\ AB^2 + BC^2 + CA^2 = 20(x^2 + y^2) + 20^2 = 2400$

$$\Rightarrow\ \frac{1}{100}(AB^2 + BC^2 + CA^2) = 24$$



11. There are several tea cups in the kitchen, some with handles and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?

Ans. (29)

Sol. Let the no. of cups with handles be x and no. of cups without handle be y

$$\binom{x}{2}\binom{y}{3} = 1200$$

As $\binom{y}{3} \Big| 1200$

$\Rightarrow\ y \le 20$ (As $\binom{21}{3} = 21 \times 20 \times 19 = 1330 > 1200$)

Also $\binom{y}{3} = \frac{y(y-1)(y-2)}{3} \Big| 1200$

$\Rightarrow\ y \ne p, p+1, p+2$, where p prime ≥ 7

$\Rightarrow$ $y \neq 7, 8, 9, 11, 12, 13, 14, 15, 17, 18, 19, 20$

$\Rightarrow$ Possible $y = 3, 4, 5, 6, 10, 16$

But $y \neq 16$ as $7 \nmid \binom{y}{3}$

After checking each value of y, we get

$y = 4, x = 25 \Rightarrow x + y = 29$

and $y = 10, x = 5 \Rightarrow x + y = 15$

and $y = 5, x = 16 \Rightarrow x + y = 21$

$\Rightarrow$ Max $(x + y) = 29$

12. Determine the number of 8-tuples $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_8)$ such that $\varepsilon_1, \varepsilon_2, ..., \varepsilon_8 \in \{1, -1\}$ and $\varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3 + ... + 8\varepsilon_8$ is a multiple of 3.

Ans. (88)

Sol. $\varepsilon_1 + 2\varepsilon_2 + 3\varepsilon_3 + 4\varepsilon_4 + 5\varepsilon_5 + 6\varepsilon_6 + 7\varepsilon_7 + 8\varepsilon_8 \equiv 0 \pmod 3$

$\Rightarrow \varepsilon_1 - \varepsilon_2 + 0 + \varepsilon_4 - \varepsilon_5 + 0 + \varepsilon_7 - \varepsilon_8 \equiv 0 \pmod 3$

$\Rightarrow \varepsilon_1 + \varepsilon_4 + \varepsilon_7 \equiv \varepsilon_2 + \varepsilon_5 + \varepsilon_8 \pmod 3$

Now $\varepsilon_i + \varepsilon_j + \varepsilon_k = 3 \equiv 0 \pmod 3 \Rightarrow$ 1 way (each of them 1)

and $\varepsilon_i + \varepsilon_j + \varepsilon_k = -3 \equiv 0 \pmod 3 \Rightarrow$ 1 way (each of them -1)

and $\varepsilon_i + \varepsilon_j + \varepsilon_k = 1 \pmod 3 \Rightarrow$ 3 ways (two of them 1 and one -1)

and $\varepsilon_i + \varepsilon_j + \varepsilon_k = -1 \pmod 3 \Rightarrow$ 3 ways (two of them -1 and one 1)

$\Rightarrow$ $\varepsilon_1 + \varepsilon_4 + \varepsilon_7 \equiv 0 \equiv \varepsilon_2 + \varepsilon_5 + \varepsilon_8 \pmod 3$ in $2 \times 2 = 4$ ways

$\varepsilon_1 + \varepsilon_4 + \varepsilon_7 \equiv 1 \equiv \varepsilon_2 + \varepsilon_5 + \varepsilon_8 \pmod 3$ in $3 \times 3 = 9$ ways

$\varepsilon_1 + \varepsilon_4 + \varepsilon_7 \equiv -1 \equiv \varepsilon_2 + \varepsilon_5 + \varepsilon_8 \pmod 3$ in $3 \times 3 = 9$ ways

Number of ways to select $(\varepsilon_1, \varepsilon_4, \varepsilon_7, \varepsilon_2, \varepsilon_5, \varepsilon_8)$ is $4 + 9 + 9 = 22$ ways

Now $\varepsilon_3, \varepsilon_6$ can be -1 or 1 $\Rightarrow$ there are $2 \times 2 = 4$ choices for $\varepsilon_3, \varepsilon_6$

$\Rightarrow$ Total number of ways to select $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_7, \varepsilon_8, \varepsilon_9)$ is $22 \times 4 = 88$ ways

13. In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the medium through A.

Ans. (24)

Sol. $[ABC] = \frac{1}{2}bc = \frac{1}{2}a \times 3$

$\Rightarrow$ $bc = 3a$ .....(i)

$[ABN] + [ANC] = [ABC]$

$\Rightarrow \frac{1}{2}c.4\sin 45° + \frac{1}{2}b.4\sin 45° = \frac{1}{2}bc$

$\Rightarrow b + c = \frac{1}{2\sqrt{2}}bc$

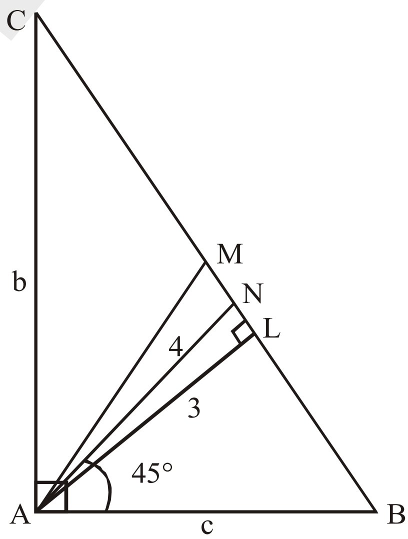
$\Rightarrow b^2 + c^2 + 2bc = \frac{1}{8}b^2c^2$

$\Rightarrow a^2 + 6a = \frac{9}{8}a^2$, (from (i))

$\Rightarrow a + 6 = \frac{9}{8}a$ (As $a \neq 0$)

$\Rightarrow a = 48$

$\Rightarrow AM = MB = MC = \frac{a}{2} = 24$



14. If x = cos1° cos2° cos3° ... cos89° and y = cos2° cos6° cos10° ... cos86°, then what is the integer nearest to $\frac{2}{7}\log_2(y/x)$?

Ans. **(19)**

Sol. $x=\prod_{r=1}^{89}\cos r^{\circ}$

$\Rightarrow \quad x=\sqrt{\prod_{r=1}^{89}\cos r^{\circ}\cos(89+1-r)}$

$\Rightarrow \quad x=\sqrt{\frac{1}{2^{89}}\prod_{r=1}^{89}\sin 2r^{\circ}}$

$\Rightarrow \quad x=\sqrt{\frac{1}{2^{89}}\left(\prod_{r=1}^{44}\sin 2r^{\circ}\right)^2.\sin 90^{\circ}}$ (As sin2(89 + 1 – r) = sin(180° – 2r) = sin2r)

$\Rightarrow \quad x=\frac{1}{2^{44}\sqrt{2}}\prod_{r=1}^{44}\sin 2r^{\circ}$

$\Rightarrow \quad x=\frac{1}{2^{44}\sqrt{2}}\sqrt{\prod_{r=1}^{44}\sin 2r\sin 2(44+1-r)}$

$\Rightarrow \quad x=\frac{1}{2^{66}\sqrt{2}}\sqrt{\prod_{r=1}^{44}\sin 4r}$

$\Rightarrow \quad x=\frac{1}{2^{66}\sqrt{2}}\sqrt{\left(\prod_{r=1}^{22}\sin 4r\right)^2}$ (As sin4(44 + 1 – r) = sin(180° – 4r) = sin4r)

$\Rightarrow \quad x=\frac{1}{2^{66}\sqrt{2}}\prod_{r=1}^{22}\sin 4r$

$=\frac{1}{2^{66}\sqrt{2}}\prod_{r=1}^{22}\sin(92^{\circ}-4r)$

$=\frac{1}{2^{66}\sqrt{2}}\prod_{r=1}^{22}\cos(4r-2)$

$\Rightarrow \quad x=\frac{1}{2^{66}\sqrt{2}}y$

$\Rightarrow \quad \frac{y}{x}=2^{66+\frac{1}{2}}\Rightarrow\frac{2}{7}\log_2\frac{y}{x}=\frac{133}{2}\times\frac{2}{7}=19$

15. Let a and b be natural numbers such that 2a – b, a – 2b and a + b are all distinct squares. What is the smallest possible value of b?

Ans. **(21)**

Sol. Let $2a-b=x^2$ ....(i)

and $a-2b=y^2$ ....(ii)

and $a+b=z^2$ ....(iii)

where $x, y, z \in \mathbb{N}_0$

Now (ii) + (iii)

$\Rightarrow$ $2a - b = y^2 + z^2$

$\Rightarrow$ $x^2 = y^2 + z^2$ ....(iv)

From (i) + (iii), $3a = x^2 + z^2$

$\Rightarrow$ $3|(x^2 + z^2)$ $\Rightarrow$ $3|x$ and $3|z$

From (iii) – (ii), $3b = z^2 - y^2$ ....(v)

$\Rightarrow$ $3|(z^2 - y^2)$ $\Rightarrow$ $3|y^2$ (as $3|z$)

$\Rightarrow$ $3|y$

$\Rightarrow$ $x = 3x_1$, $y = 3y_1$, $z = 3z_1$

$\Rightarrow$ $x_1^2 = y_1^2 + z_1^2$ .....(vi) (from (iv))

Let as assume every two of z_1, y_1, x_1 are coprime $\Rightarrow$ z_1, y_1, x_1 is a primitive Pythagorean triplet

$\Rightarrow$ out of y_1 and z_1 one even ≥ 4 and other odd ≥ 3

From (v), $b = 3(z_1^2 - y_1^2) = 3(z_1 + y_1)(z_1 - y_1)$

Now we need $z_1 + y_1$ and $z_1 - y_1$ as small as possible $\Rightarrow$ $z_1 = 4$, $y_1 = 3$ $\Rightarrow$ $x_1 = 5$

$\Rightarrow$ min $b = 3 \times (4 + 3)(4 - 3) = 21$

16. What is the value of

$$\sum_{\substack{1\le i<j\le 10\\ i+j=\text{odd}}} (i+j) - \sum_{\substack{1\le i<j\le 10\\ i+j=\text{even}}} (i+j)?$$

Ans. **(55)**

Sol. $S = \sum_{\substack{1\le i<j\le 10\\ i+j=\text{odd}}} (i+j) - \sum_{\substack{1\le i<j\le 10\\ i+j=\text{even}}} (i+j)?$

$$\Rightarrow S = \sum_{1\le i<j\le 10} (-1)^{i+j-1}(i+j)$$

$$\Rightarrow S = \frac{1}{2}\sum_{1\le i<j\le 10} (-1)^{i+j-1}(i + j + 22 - (i + j))$$

$$\Rightarrow S = 11 \sum_{1\le i<j\le 10} (-1)^{i+j-1}$$

Now let us count how many times i + j is even or odd

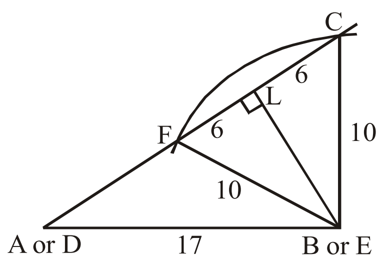
For i + j = even, there are $2.{}^5C_2 = 20$ terms

For i + j = odd, there are ${}^5C_1.{}^5C_1 = 25$ terms

$\Rightarrow S = 11(-20 + 25) = 55$

17. Triangles ABC and DEF are such that $\angle A = \angle D$, $AB = DE = 17$, $BC = EF = 10$ and $AC - DF = 12$. What is $AC + DF$?

Ans. **(30)**



Sol. $BL = \sqrt{FB^2 - FL^2} = \sqrt{10^2 - 6^2} = 8$

DL or $AL = \sqrt{AB^2 - BL^2} = \sqrt{17^2 - 8^2} = 15$

Now $DF + AC = (DL - FL) + (AL + LC)$

$= 2AL = 2 \times 15 = 30$

18. If a, b, c ≥ 4 are integers, not all equal and $4abc = (a + 3)(b + 3)(c + 3)$, then what is the value of a + b + c?

Ans. **(16)**

Sol. $4abc = (a + 3)(b + 3)(c + 3)$

$$\Rightarrow \left(1+\frac{3}{a}\right)\left(1+\frac{3}{b}\right)\left(1+\frac{3}{c}\right) = 4$$

W.L.O.G. let $4 \le a \le b \le c$

$$\Rightarrow \frac{1}{a} \ge \frac{1}{b} \ge \frac{1}{c} \Rightarrow 1+\frac{1}{a} \ge 1+\frac{3}{b} \ge 1+\frac{3}{c}$$

So $\left(1+\frac{3}{a}\right)^3 \ge 4 \Rightarrow 1+\frac{3}{a} \ge 4^{\frac{1}{3}}$

$$\Rightarrow a \le \frac{3}{4^{\frac{1}{3}}-1} = 4^{2/3}+1+4^{1/3} < 3+1+2$$

$\Rightarrow$ a < 6 $\Rightarrow$ a = 4 or 5

for a = 5, $\left(1+\frac{3}{b}\right)\left(1+\frac{3}{c}\right) = \frac{5}{2}$

$$\Rightarrow \left(1+\frac{3}{b}\right)^2 \ge \frac{5}{2}$$

$$\Rightarrow b \le \frac{3}{\left(\frac{5}{2}\right)^{1/2}-1} = 2\left(\left(\frac{5}{2}\right)^{1/2}+1\right) < 2(2+1) = 6$$

$\Rightarrow b \le 5 \Rightarrow b = 5$ (as b ≥ a)

$$\Rightarrow 1+\frac{3}{c} = \frac{25}{16} \Rightarrow c = \frac{16}{3} \notin \mathbb{Z} \Rightarrow a \ne 5$$

For a = 4, $\left(1+\frac{3}{b}\right)\left(1+\frac{3}{c}\right) = \frac{16}{7}$

$$\left(1+\frac{3}{b}\right)^2 \ge \frac{16}{7} \Rightarrow b \le \frac{3}{\frac{4}{\sqrt{7}}-1} = \frac{7}{3}\left(\frac{4}{\sqrt{7}}+1\right) < 6$$

$\Rightarrow$ b = 4 or 5

for b = 4, $c = \frac{49}{5} \notin \mathbb{Z}$

for b = 5, c = 7 $\Rightarrow$ a + b + c = 4 + 5 + 7 = 16

19. Let N = 6 + 66 + 666 +...+ 666 ... 66, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N?

Ans. (33)

Sol. $N = 6 + 66 + 666 + ... + \underbrace{6666......6}_{100}$

$$= \frac{6}{9}\left[10+10^2+.......+10^{100}-100\right]$$

$$= \frac{6}{9}\left(\frac{10\left(10^{100}-1\right)}{9}-100\right)$$

$$= \frac{20}{3}\left(\frac{999.....9}{9}-10\right)$$

$= \frac{20}{3}\left[\underbrace{111.....1}_{98 \text{ times}}01\right]$

$= \frac{222....2020}{3}$

$= \frac{222\times10^{98} + 222\times10^{95} + ... + 222\times10^{5}}{3} + \frac{22020}{3}$

$= \underbrace{74\times10^{98} + 74\times10^{95} + ... + 74\times10^{5}}_{32 \text{ terms}} + 7340$

$= \underbrace{740740 \text{ ...... } 740}_{32 \text{ Blocks of } 740}\ 7340$

$\Rightarrow$ 33 sevens

20. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.

Ans. (17)

Sol. Let product of digits of n be P(n)

Claim : $P(n) \le n$

Proof: Let $n = a_m 10^m + a_m\ 10^{m-1} + ... + a_0 \ge a_m 10^m \ge a_m 9^m \ge a_m a_{m-1} \text{ ... } a_0$

$\Rightarrow\ n \ge P(n)$

Now $n^2 - 15n - 27 \le n$

$\Rightarrow\ n^2 - 16n - 27 \le 0 \ \Rightarrow\ (n-8)^2 \le 91$

$\Rightarrow\ n \le 8 + \sqrt{91} < 18$ ...(i)

Also $P(n) \ge 0$

$\Rightarrow\ n^2 - 15n - 27 \ge 0$

$\Rightarrow\ n^2 - 15n + 56 \ge 83$

$\Rightarrow\ (n-7)(n-8) \ge 83$

$\Rightarrow\ (n-7)^2 > (n-7)(n-8) \ge 83$

$\Rightarrow\ (n-7) > \sqrt{83}$

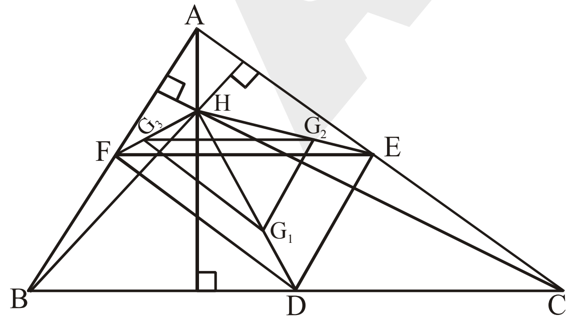
$\Rightarrow\ n > 7 + \sqrt{83} > 16$ ...(ii)

From (i) and (ii) n = 17, which satisfies the given condition $\Rightarrow$ Required sum = 17.

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2 and G_3 be the centroids of the triangles HBC, HCA and HAB, respectively. If the area of triangle $G_1G_2G_3$ is 7 units, what is the area of triangle ABC?

Ans. (63)

Sol.



Given

$\frac{HG_1}{HD} = \frac{HG_2}{HE} = \frac{2}{3}$

$\Rightarrow$ $G_1G_2 \parallel DE$

Similarly $G_1G_3 \parallel DF$ and $G_2G_3 \parallel FE$

$\Rightarrow$ $[HG_1G_2] = \frac{4}{9}[HDE]$ ... (1)

and $[HG_1G_3] = \frac{4}{9}[HFD]$ ... (2)

and $[HG_2G_3] = \frac{4}{9}[HFE]$ ... (3)

From (1) + (2) – (3), we get,

$$[HG_1G_2] + [HG_1G_3] - [HG_2G_3] = \frac{4}{9}([HDE] + [HFD] - [HFE])$$

$\Rightarrow$ $[G_1G_2G_3] = \frac{4}{9}[DEF]$

$\Rightarrow$ $4[DEF] = 9[G_1G_2G_3]$

$\Rightarrow$ $[ABC] = 9 \times 7 = 63$ (as ΔDEF is median triangle of ΔABC)

22. A positive integer k is said to be good if there exists a partition of {1, 2, 3, ... 20} in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there?

Ans. (06)

Sol. Let us partition it in n part and each part has sum = k then,

$nk = 1 + 2 + 3 + ... + 20$

$\Rightarrow nk = 210$

$\Rightarrow k \mid 210$

Also k must be ≥ 20, (as 20 will be present in some partition)

Now, $210 = 2 \times 3 \times 5 \times 7$

So, Proper divisors of 210 are 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105

$\Rightarrow$ k can be 21, 20, 35, 42, 70, 105

For k = 21, we have (1, 20), (2, 19), ... (10, 11)

$\Rightarrow$ 21 is good number

For k = 42, join two-two pairs of above

For k = 105, join five-five pairs of above

$\Rightarrow$ 42 and 105 are also good numbers.

For k = 30, we have

{20, 10}, {19, 11}, {18, 12}, {17, 13}, {16, 14}, {15, 9, 6}, {1, 2, 3, 4, 5, 7, 8}

$\Rightarrow$ k = 30 is also a good number

For k = 35, we have

{5, 9, 11, 10} {6, 7, 8, 14}, {4, 15, 16}, {17, 18}, {2, 13, 20}, {1, 3, 12, 19}

k = 35 is also a good number.

For k = 70

Join two-two pairs of above

$\Rightarrow$ k = 70 is a good number

Hence, there are total 6 good number.

23. What is the largest positive integer n such that $\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n(a+b+c)$ holds for all positive real numbers a, b, c

Ans. **(14)**

Sol. We know that for a, b, c real numbers and x, y, z positive reals, we have

$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}$; where equality holds for $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$

$$\therefore \quad \frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq \frac{(a+b+c)^2}{\frac{b}{29}+\frac{c}{31}+\frac{c}{29}+\frac{a}{31}+\frac{a}{29}+\frac{b}{31}} = \frac{(a+b+c)^2}{\frac{a+b+c}{29}+\frac{a+b+c}{31}}$$

$$= \frac{(a+b+c)^2}{(a+b+c)\left[\frac{1}{29}+\frac{1}{31}\right]}$$

$$= (a+b+c)\left(\frac{899}{60}\right)$$

$$= \left(14+\frac{59}{60}\right)(a+b+c)$$

$\geq n\,(a+b+c)$

$\Rightarrow$ Largest positive integer n = 14

24. If N is the number of triangles of different shapes (i.e. not similar) whose angle are all integers (in degrees), what is N/100?

Ans. **(27)**

Sol. Let the angles be λ_1, λ_2, λ_3

$\lambda_1 + \lambda_2 + \lambda_3 = 180°$

Number of positive solution are ${}^{180-1}C_{3-1} = {}^{179}C_2$

But some solutions are counted more than once like,

1 1 178
2 2 176
3 3 174
:
59 59 62
61 61 58
62 62 56
:
89 89 2

} these are 88 solutions each of these are counted 3 times

Every solution with $\lambda_1 \neq \lambda_2 \neq \lambda_6$ is counted 6 times.

$\lambda_1 = \lambda_2 = \lambda_3 = 60$ counted only once

$$\Rightarrow \quad N = \underbrace{\frac{1}{6}\left({}^{179}C_2 - 3\times 88 - 1\right)}_{\text{scalene}} + \underbrace{88}_{\text{isosceles but not equilateral}} + \underbrace{1}_{\text{equilateral}}$$

$$\Rightarrow \quad N = 2700 \Rightarrow \frac{N}{100} = 27$$

25. Let T be the smallest positive integer which, when divided by 11, 13, 15 leaves remainders in the sets {7, 8, 9}, {1, 2, 3}, {4, 5, 6} respectively. What is the sum of the squares of the digits of T?

Ans. **(81)**

Sol. LCM (11, 13, 15) = 2145

LCM (13, 15) = 195

LCM (11, 15) = 165

LCM (11, 13) = 143

Let us consider $y_1, y_2, y_3 \in \mathbb{N}_0$ such that $195\, y_1 \equiv 1 \pmod{11}$; $165\, y_2 \equiv 1 \pmod{13}$ and $143\, y_3 \equiv 1 \pmod{15}$

$\Rightarrow$ $y_1 \equiv 7$ (mode 11); $y_2 \equiv 3 \pmod{13}$ and $y_3 \equiv 2 \pmod{15}$

Now using Chinese remainder theorem, we get

$T \equiv a_1 y_1\, 195 + b_1 y_2 165 + c_1 y_3 143 \pmod{2145}$

where $a_1 \in \{7, 8, 9\}$, $b_1 \in \{1, 2, 3\}$, $c_1 \in \{4, 5, 6\}$

Let $a_1 = a + 8$, $b_1 = b + 2$, $c_1 = c + 5$

where a, b, c $\in \{-1, 0, 1\}$

$T \equiv 1365(a + 8) + 495(b + 2) + 286(c + 5) \pmod{2145}$

$T \equiv 13340 + 1365a + 495b + 286c \pmod{2145}$

or $T \equiv 470 - 780a + 495b + 286c \pmod{2145}$

Now we can see that min $T \leq 470$

for $a = -1$ and any choice of $b \neq -1$, $c \neq -1$ we get $T > 470$

for $a = b = c = -1$, $T = 469$ $\Rightarrow$ $T \leq 469$

for $a = 1$ and $b \neq 1$, $T > 469$

for $a = 1$, $b = 1$, $T \equiv 185 + 286c \pmod{2145}$

$\Rightarrow$ $T \leq 185$ (for $c = 0$ equality)

Finally $a = 0$, $T \equiv 470 + 495b + 286c \pmod{2145}$

$T \equiv 470 + 495b + 286c \pmod{2145}$

for $b = 0$, $c = -1$, $T \equiv 184 \pmod{2145}$

$\Rightarrow$ $T \leq 184$ (Equality for $a = 0$, $b = 0$, $c = -1$)

In case of $a = 0$ and $(b, c) \neq (0, -1)$, $T > 184$

We get smallest T = 184

Now required sum = $1^2 + 8^2 + 4^2 = 81$

Aliter :

$T \equiv \{4, 5, 6\} \pmod{15}$

or $T \equiv$ {19,20,21},{34,35,36},{49,50,51},{64,65,66},{79,80,81},{94,95,96},{109,110,111},
{124,125,126},{139,140,141},{154,155,156},{169,170,171},{184,185,186} (mod 15)

Now by direct checking we get smallest

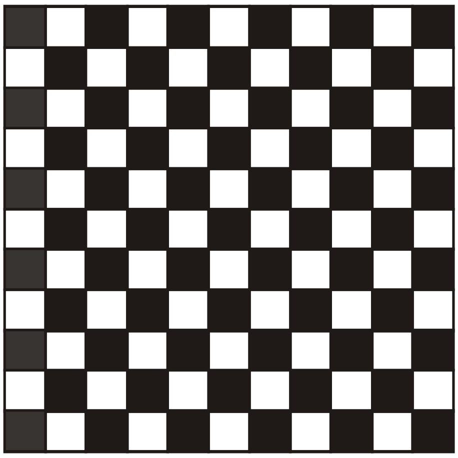
T = 184

$\Rightarrow$ Required sum = $1^2 + 8^2 + 4^2 = 81$

26. What is the number of ways in which one can choose 60 unit squares from a 11 × 11 chessboard such that no two chosen squares have a side in common?

Ans. (62)

Sol.



Let us colour each unit square alternatively as black and white there will be 61 non adjecent black squares and 60 non adjacent white squares

We can't select 60 non adjacent squares in combination of black and white both. As if we select one black, atleast 2 white we can't select. Suppose we select k black then atleast k + 1 whites (adjacent to these blacks) we can't select implies we left with atmost 59 – k white squares and we need 60 – k white which is not possible!

So in order to selected 60 non adjacent squares we need to select all black or all white this can be done in $\binom{61}{60}+\binom{60}{60}=61+1=62$ ways

27. What is the number of ways in which one can colour the squares of a 4 × 4 chessboard with colous red and blue such that each row as well as each column has exactly two red squares and two blue squares?

Ans. (90)

Sol. Each row can be coloured in any one of the following six ways

RBRB, RRBB, RBBR, BBRR, BRRB and BRBR

First two rows can be coloured in 6 × 6 = 36 ways

Let us divide all 36 ways in three cases.

(i) First and second row are identical :

There are 6 such cases then last two rows can be painted in only 1 way.

⇒ Number of such ways = 6

(ii) First and second row do not match at any place :

Colour first row by any one of the 6 ways and switch colour in second row for corresponding squares.

⇒ First two can be coloured in 6 ways.

Now 3^{rd} row can be painted in any one of the 6 ways and final row in one way.

$\Rightarrow$ $6 \times 6 = 36$ ways in this case

(iii) First and second row match exactly at two places:

There are $36 - 6 - 6 = 24$ such cases

The column in which two squares are of same colour (in first two row) can be painted in only one way (in third and fourth row) and the remaining two squares of third row can be painted in two ways then last row will be in one way.

$\Rightarrow 24 \times 2 = 48$ ways in this case.

Hence total ways are $6 + 36 + 48 = 90$ ways.

28. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N.

Ans. (24)

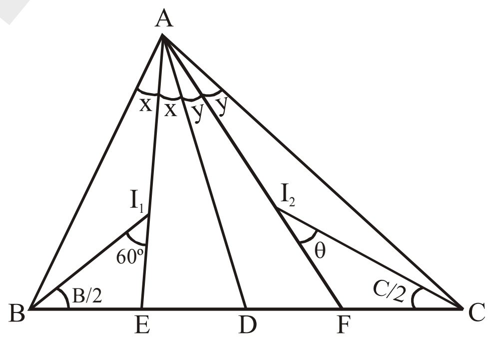
Sol. There are two ways to partition 8 in unequal size which are 1 + 2 + 5 and 1 + 3 + 4.

Hence total ways of distribution = $\left(\frac{8!}{1!2!5!} + \frac{8!}{1!3!4!}\right) \times 3! = 2688$ ways

$\Rightarrow$ sum of digits = 2 + 6 + 8 + 8 = 24

29. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60°$, what is the measure of $\angle CI_2F$ in degrees?

Ans. (30)



Sol. Let $\angle CI_2F = \theta$, $\angle BAE = x = \angle EAD$

and $\angle DAF = y = \angle FAC$

$\Rightarrow \angle A = 2x + 2y$ or $x + y = \frac{\angle A}{2}$

$\Rightarrow$ $\angle EAF = \frac{\angle A}{2}$

Now $\angle AEF = \frac{\angle B}{2} + 60°$

and $\angle AFE = \theta + \frac{\angle C}{2}$

In ΔAEF,

$$\left(\frac{\angle B}{2} + 60°\right) + \left(\theta + \frac{\angle C}{2}\right) + \frac{\angle A}{2} = 180°$$

or $\theta + \frac{\angle A + \angle B + \angle C}{2} + 60° = 180°$

$\Rightarrow$ $\theta = 180° - 150° = 30°$

i.e. $\angle CI_2F = 30°$

30. Let $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a polynomial in which a_i is a non negative integer for each $i \in (0, 1, 2, 3, ..., n)$. If $P(1) = 4$ and $P(5) = 136$, what is the value of $P(3)$?

Ans. **(34)**

Sol. $P(5) = a_0 + 5a_1 + 25a_2 + 125a_3 + 625a_4 + ... + a_n5^n = 136$

as each $a_1 \in \mathbb{N}_0$, if any $a_i \geq 1$ for $i \geq 4$, then LHS > 136

$\Rightarrow a_4 = a_5 = a_6 = ... = a_n = 0$

$\Rightarrow a_0 + 5a_1 + 25a_2 + 125a_3 = 136$ ......(1)

Also a_3 can be 0 or 1 only

Now $P(1) = a_1 + a_1 + a_2 + a_3 = 4$ ......(2)

$\Rightarrow a_0, a_1, a_2, a_3 \in \{0, 1, 2, 3, 4\}$

If $a_3 = 0$, then

$a_0 + 5a_1 + 25a_2 \leq 4 + 20 + 100 = 124 < 136$

$\Rightarrow a_3 = 1$

$\Rightarrow a_0 + 5a_1 + 25a_2 = 11$ (from (1))

$\Rightarrow a_2 = 0$

$\Rightarrow a_0 + 5a_1 = 11$ ......(3)

Also from (2) $a_0 + a_1 = 3$ ......(4)

$\Rightarrow a_1 = 2, a_0 = 1$

Hence $P(x) = 1 + 2x + x^3$

$\Rightarrow P(3) = 1 + 6 + 27 = 34$