34th Indian National Mathematical Olympiad-2019

Date of Examination : 20th January, 2019

SOLUTIONS

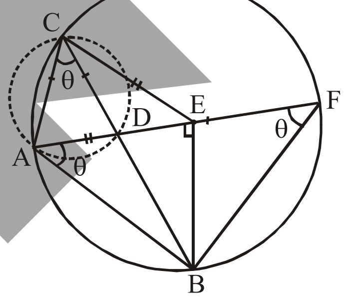
1. Let ABC be a triangle with $\angle BAC > 90°$. Let D be a point on the segment BC and E be a point on the line AD such that AD is tangent to the circumcircle of triangle ACD at A and BE is perpendicular to AD. Given that CA = CD and AE = CE, determine $\angle BCA$ in degrees.

Sol. Construction : Extend

AE to meet the circumcircle of ΔABC at F

Claim : We prove that E is the circumcentre of ΔABC

Let $\angle BAD = \theta$ then $\angle ACD = \theta$ (By alternate segment theorem)



Join $\overline{BF}$, we have

$\angle BCA = \angle AFB = \theta$ (Angle made by $\overline{AB}$)

$\therefore$ We have AB = BF $\Rightarrow$ ΔABF is isosceless

and $BE \perp AF \Rightarrow$ E is mid point of AF

$\therefore$ AE = EF and AE = CE (given)

$\therefore$ AE = EF = CE

$\Rightarrow$ E is the circum centre of ΔABC

$\therefore$ $\angle ABF = 90º$ [AF is diameter]

$\therefore$ $180 - 2\theta = 90º$

$90º = 2\theta$

$\theta = 45º$

$\therefore$ $\angle BCA = 45º$

2. Let $A_1B_1C_1D_1E_1$ be a regular pentagon. For $2 \le n \le 11$, let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoints of the sides of $A_{n-1}\, B_{n-1}\, C_{n-1}\, D_{n-1}\, E_{n-1}$. All the 5 vertices of each of the 11 pentagons are arbitrarily coloured red or blue. Prove that four points among these 55 points have the same colour and form the vertices of a cyclic quadrilateral.

Sol. Let P_i be the polygon $A_iB_iC_iD_iE_i$ and O be the centre of the polygon

$\Rightarrow$ $P_1, P_3, P_7, P_9, P_{11}$ have the same orientation w.r.t. O

Let C_i be the colour dominating in polygon P : [Which means which has atleast 3 of same colour]

Now in $P_1, P_3, P_7, P_9, P_{11}$, atleast three will have colour with same P_i

Let them be $P_{1_1}, P_{1_2}, P_{1_3}$

Now the C_i for these three be red (W. L. O. G.)

Now P_{1_1} has 3 vertices of of same colour let them be $V_{1_1}, V_{1_2}, V_{1_3}$

Compare the vertices of P_{1_1}, P_{1_2}

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If any of V_{2_4}, V_{2_5} is not red then $\exists$ 2 of V_{2_1}, V_{2_2}, V_{2_3}

which are red if they are V_{1_1}, V_{1_2} then

V_{1_1}, V_{1_2}, V_{2_1}, V_{2_2} is cyclic

so V_{2_4}, V_{2_5} should be red

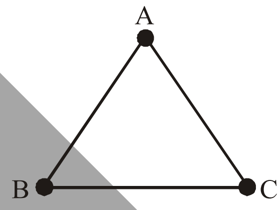
similarly V_{3_4}, V_{3_5} are red

Now we got V_{2_4}, V_{2_5}, V_{3_4}, V_{3_5} are cyclic and red

So we can find a cyclic quadrilateral.

Solution 2 : Consider a regular triangle in the plane ABC, whose vertices are coloured using only two colours Red and Blue.

then by PHP, two of the vertices must have same colour (say A & B)



Now if we consider a regular pentagon then by using above result, we can assure that 3 vertices of pentagon is in Red and 2 vertices are blue and vice versa.

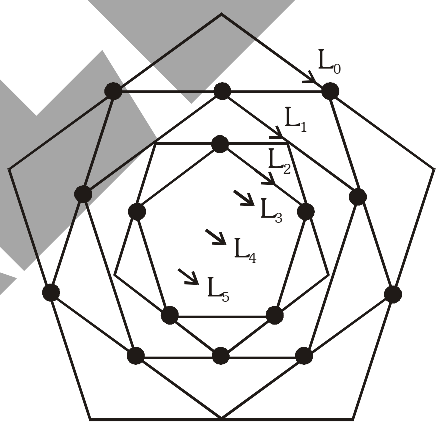
Now, we will analyse cases by case

Case-1 : When 4th vertices will have some colour as 3 vertices, then we are done, as any 4 vertices of a regular pentagon will form a cyclic quadrilateral having all the 4 vertices same coloured.

Case-2 : Any of the 3 vertices Red and 2 are Blue coloured

Now, observe that in any pentagon two of the vertices will have same colour and it is also clear that, for this case we will have 6 pentagons (i.e. 1st, 3rd, 5th, 7th, 9th and 11th pentagon).

So, we have 6 such pantagon in which we will get 5 set of parallel lines (parallel to original pentagon)



Here L_1, L_2, L_3, L_4, L_5 are 5 set of parallel lines

So, once we consider there 5 set of parallel lines and we have six such lines (including L_0) then by PHP, We will get 4 points having same colour and an isosceles trapezium which will be cyclic and hence we are done.

Case-3 : 2R and 3B : Same arguments as of case 2.

3. Let m, n be distinct positive integers.

Prove that $\gcd(m, n) + \gcd(m + 1, n + 1) + \gcd(m + 2, n + 2) \le 2|m - n| + 1$.

Further, determine when equality holds.

Sol. let $m > n$

$\gcd(m, n) = \gcd(m, m - n) = a$

$\gcd(m + 1, n + 1) = \gcd(m + 1, m - n) = b$

$\gcd(m + 2, n + 2) = \gcd(m + 2, m - n) = c$

$\Rightarrow \gcd(a, b) = 1, \gcd(b, c) = 1, \gcd(a, c) / 2$

Case-1 :

If $\gcd(a, c) = 1$, $d = m - n$

a/d, b/d, c/d

$\Rightarrow abc/d$

$\Rightarrow d \ge abc$

$\Rightarrow 2d + 1 \ge 2abc + 1$

if atleast one of a, b, c > 1 let it be b

$$\begin{aligned}\Rightarrow 2abc + 1 &= abc + abc + 1\\ &\ge 2ac + dbc + 1\\ &\ge ac + ac + b + 1\\ &\ge a + c + b + 1\\ &> a + b + c\end{aligned}$$

so we are done

if all of a, b, c = 1

then

$2abc + 1 = 3 = a + b + c$

so we are done

Case-2 : $\gcd(a, c) = 2$

Now $a = 2a'$, $c = 2c' \Rightarrow \gcd(a'c') = 1$

$\Rightarrow 2a'bc'/d \Rightarrow d \ge 2a'bc'$

Now $\quad 2d + 1 \ge 4a'bc' + 1$

if $\quad b \ge 2$

then
$$\begin{aligned}4a'b'c + 1 &\ge 2a'c' + 1 + 2a'c'b\\ &\ge 4a'c' + 1 + 2b\\ &\ge 2a' + 2c' + 1 + 2b\\ &> 2a' + 2c' + b = a + b + c\end{aligned}$$

if $b = 1$ $\quad 2d + 1 \ge 4a'c' + 1 \ge 2a' + 2c' + b = a + b + c$

so equality holds when $a = b = c = 1 \Rightarrow a' = c' = 1, b = 1$

$d = abc$

which means at

$|m - n| = 1$; for m, n consecutive positive integers

$|m - n| = 2$ and m,n are even positive integers

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4. Let n and M be positive integers such that $M > n^{n-1}$. Prove that there are n distinct primes $p_1, p_2, p_3 \ldots, p_n$ such that p_j divides M + j for $1 \le j \le n$.

Sol. $n, m \in I^+$,

$M > n^{n-1}$, $n \rightarrow$ distinct primes

$P_1, P_2, \ldots\ldots P_n$ such that p_j divides M + j for $i \le j \le n$

Case-1 : If M + j has atleast 'n' prime divisiors then p_j divides m + j

for $i \le j$

for at least 'n' distinct primes.

Case-2 : When m + j has n – 1 or less prime divisors,

Let $M + j = P_1^{m_1}.P_2^{m_2}\ldots\ldots P_t^{m_t}$ where $P_1, P_2 \ldots\ldots P_t$ are main distinct $t \le n - 1$

Claim :

Let us assume that for P_i, P^{n_i} is maximum,

Suppose 'P' is chosen for M + i & M + j & P^m & P^n divides M + i & M + j

when $n \ge m$

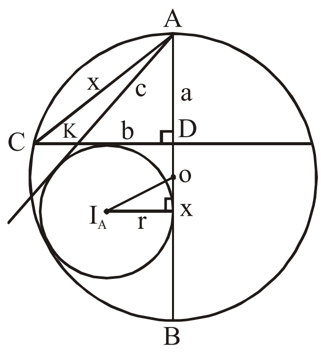
$\Rightarrow$ P^m divides $(m + j) - (m + i) = j - i \le n - 1$

$$\text{but } P^m \ge (m+j)^{\frac{1}{n-1}} \ge \left(n^{n-1}\right)^{\frac{1}{n-1}} = n$$

Which leads to a contradiction.

5. Let AB be a diameter of a circle Γ and let C be a point on Γ different from A and B. Let D be the foot of perpendicular from C on to AB. Let K be a point of the segment CD such that AC is equal to the semiperimeter of the triangle ADK. Show that the excircle of triangle ADK opposite A is tangent to Γ.

Sol. Since if two circles touch each other then difference between their centre is the difference between their radii if two circle touch internally



Our aim is to show $OI_A = R - r$

Let AD = a, AK = c, KD = b

let AC = x, $XI_A = r$, $x = \dfrac{a+b+c}{2}$

Since exradius ΔADK is $r = \dfrac{KD + AK - AD}{2}$

$$\therefore \quad r = \frac{b+c-a}{2} = \frac{b+c+a-2a}{2} = \frac{b+c+a}{2} - a$$

$$\boxed{r = x - a}$$

Also $\quad AC^2 = AD \times AB$

$x^2 = 2aR \quad \ldots\ldots (A)$

Also $\quad OX = |AD + DX - AO| = |a + r - R|$

$OX = |x - R|$

In ΔOI_AX $\quad \Rightarrow OI_A^2 = OX^2 + XI_A^2$

$= (x - R)^2 + r^2$

$= x^2 + R^2 - 2xR + r^2$

$= R^2 + r^2 + 2aR - 2xR$, from (A)

$R^2 + r^2 - 2R(x - a) = R^2 + r^2 - 2rR$

$OI_A^2 = (R - r)^2$

$\therefore OI_A = R - r$

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6. Let f be a function defined from the set $\{(x, y) : x, y \text{ real}, xy \neq 0\}$ to the set of all positive real numbers such that

(i) $f(xy, z) = f(x,z)f(y,z)$, for all $x, y \neq 0$;

(ii) $f(x, 1 - x) = 1$, for all $x \neq 0, 1$.

Prove that

(a) $f(x, x) = f(x, -x) = 1$, for all $x \neq 0$;

(b) $f(x, y)f(y, x) = 1$, for all $x, y \neq 0$.

Sol. Given information is insufficient to prove the required results. One such counter example is as follows.

Counter example :

$$f(x, y) = \begin{cases} g(x) & , \quad y = c \\ 1 & , \quad y \neq c \end{cases}$$

g is some multiplicative function such that $g(1 - c) = 1$

Now, $g(xy) = g(x).g(y)$

$\Rightarrow f(xy, z) = f(x, z).f(y, z)$

$$f(x, 1 - x) = \begin{cases} g(x) & , \quad x = 1 - c \\ 1 & , \quad x \neq 1 - c \end{cases} = 1$$

One such $g(x) = x^2$, and take $c = 2$, where c is some real number. Here

for $1 - x \neq c$, $f(x, 1 - x) = 1$

and for $1 - x = c$ or $x = 1 - c$, $f(x, 1 - x) = f(1 - c, c) = g(1 - c) = 1$

Hence $f(x, 1 - x) = 1$

Now observe:

$f(2, 2) = g(2) = 2^2 \neq 1$

Also for $x = y = 2$

$f(x, y)\, f(y, x) = (f(2, 2))^2 = (g(2))^2 = 16 \neq 1$

Hence the given question is incorrect.