SOLUTION
THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
KAPREKAR CONTEST - FINAL - SUB JUNIOR
CLASS - VII & VIII

Instructions:
1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

1. A lucky year is one in which at least one date, when written in the form day/month/year, has the following property: The product of the month times the day equals the last two digits of the year. For example, 1956 is a lucky year because it has the date 7/8/56 where $7 \times 8 = 56$, but 1962 is not a lucky year as $62 = 62 \times 1$ or 31×2, where 31/2/1962 is not a valid date. From 1900 to 2018 how many years are not lucky (not including 1900 and 2018)? Given proper explanation for you answer.

Sol. As maximum available date is 31 therefore, years having last two digits as prime numbers greater than 31 or multiple of prime numbers greater than 31 will not be lucky.

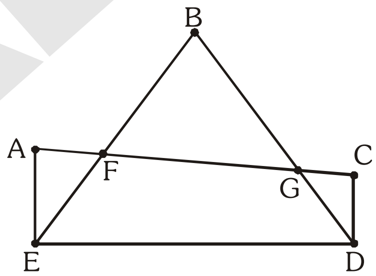
Also, 1958 and 1962 will not be lucky because 29/2/58 and 31/1/62 are not valid dates

Also, 2000 is not a lucky year because date and months are natural numbers so their product will not be 00.

1937	1958	1971	1983
1941	1959	1973	1986
1943	1961	1974	1989
1947	1962	1979	1994
1953	1967	1982	1997
2000			

Total not lucky numbers are 21.

2. In the figure given, $\angle A$, $\angle B$ and $\angle C$ are right angles. If and $\angle AEB = 40°$ and $\angle BED = \angle BDE$, then find $\angle CDE$.



Sol. In ΔBED

$x + x + 90° = 180°$

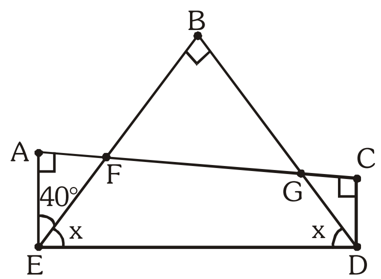
$\Rightarrow x = 45°$

In quadrilateral AEDC

$\angle CAE + \angle AED + \angle CDE + \angle ACD = 360°$

$90° + 45° + 40° + \angle CDE + 90° = 360°$

$\Rightarrow \angle CDE = 95°$



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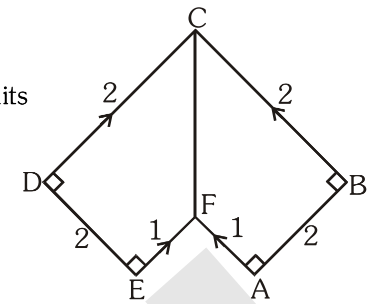
3. (a) ABCDEF is a hexagon in which AB = BC = CD = DE = 2 and EF = FA = 1. Its interior angle C is between 90° and 180° and F is greater than 180°. The rest of the angles are 90° each. What is its area?

(b) A convex polygon with 'n' sides has all angles equal to 150°, except one angle. List all possible values of n.

Sol. (a) In hexagon ABCDEF

Two trapeziums CDEF and ABCF are formed

area = $\frac{1}{2}(1 + 2)\,2 + \frac{1}{2}(1 + 2)\,2 = 3 + 3 = 6$ units



(b) Let y be the different angle

$0 < y < 180$

$0 < (n - 2)\,180 - 150\,(n - 1) < 180$

$0 < 30n - 210 < 180$

$210 < 30n < 390$

$7 < n < 13$

as n is a natural number therefore n = 8, 9, 10, 11, 12

but when n = 12, y = 150°

therefore only 4 values of n are possible

4. a, b, c are distinct non-zero reals such that $\frac{1+a^3}{a} = \frac{1+b^3}{b} = \frac{1+c^3}{c}$. Find all possible values of $a^3 + b^3 + c^3$.

Sol. $\frac{1+a^3}{a} = \frac{1+b^3}{b} = \frac{1+c^3}{c} = \frac{(1+b^3)-(1+c^3)}{b-c} = \frac{(1+c^3)-(1+a^3)}{c-a}$

$\Rightarrow \frac{b^3-c^3}{b-c} = \frac{c^3-a^3}{c-a}$

$\Rightarrow \frac{(b-c)(b^2+c^2+bc)}{b-c} = \frac{(c-a)(c^2+a^2+ac)}{c-a}$

$\Rightarrow b^2 + c^2 + bc = c^2 + a^2 + ac$

$\Rightarrow b^2 - a^2 + bc - ac = 0$

$\Rightarrow (b - a)(a + b + c) = 0$

$\Rightarrow a + b + c = 0 \qquad (\because b \neq a)$

Now, $\frac{1+a^3}{a} = \frac{1+b^3+1+c^3}{b+c}$

$\Rightarrow \frac{1+a^3}{a} = \frac{2+b^3+c^3}{-a} \quad (\because a + b + c = 0)$

$\Rightarrow 1 + a^3 = -2 - b^3 - c^3$

$\Rightarrow a^3 + b^3 + c^3 = -3$

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5. Find the smallest positive integer such that it has exactly 100 different positive integer divisors including 1 and the number itself.

Sol. Let the number be $N = P_1^{m_1} \cdot P_2^{m_2} \cdot .... P_k^{m_k}$ here P_1, P_2 ..... P_k are the prime divisors.

So, number of divisions = $(m_1 + 1)(m_2 + 1)$ .... $(m_k + 1)$. We need to find smallest N, such that $(m_1 + 1)(m_2 + 1)$ .... $(m_k + 1) = 100$.

Now, lets break 100 into product of decreasing numbers.

i.e., $100 = 100 \times 1, 50 \times 2, 25 \times 4, 20 \times 5, 10 \times 10,$

$25 \times 2 \times 2, 10 \times 5 \times 2, 5 \times 5 \times 4, 5 \times 5 \times 2 \times 2$

Now, for the number to be smallest. It should be

$N = 2^{99}, 2^{49} \times 3, 2^{24} \times 3^3, 2^{19} \times 3^4, 2^9 \times 3^9, 2^{24} \times 3 \times 5, 2^9 \times 3^4 \times 5, 2^4 \times 3^4 \times 5^3, 2^4 \times 3^4 \times 5 \times 7$

Now, amongst all of these numbers

$N = 2^4 \times 3^4 \times 5 \times 7 = 45360$, gives the least value

So, N = 45360

6. (a) What is the sum of the digits of the smallest positive integer which is divisible by 99 and has all of its digits equal to 2?

(b) When 270 is divided by the odd number n, the quotient is a prime number and the remainder is 0. What is n ?

Sol. (a) The number should be a multiple of 11 and 9.

22222.... n times

For divisibility of 11, difference of sum of digits at even and odd places should be 0,

$\therefore$ n should be even

For divisibility of 9, sum should be divisible by 9

As n is both even and multiple of 9 therefore minimum value of n will be 18

So sum of digits = $2 \times 18 = 36$

(b) Let the prime number be p

then $270 = n \times p$

as 270 is even and n is odd therefore p should be even.

Therefore p = 2

$\Rightarrow n = \frac{270}{2} = 135$

7. Consider the sums

$$A = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{99 \cdot 100} \text{ and } B = \frac{1}{51 \cdot 100} + \frac{1}{52 \cdot 99} + \ldots + \frac{1}{100 \cdot 51}$$

Express $\frac{A}{B}$ as an irreducible fraction.

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Sol. $A = \frac{1}{1.2}+\frac{1}{3.4}+\ldots+\frac{1}{99.100}$

$= \frac{2-1}{1.2}+\frac{4-3}{3.4}+\ldots+\frac{100-99}{99.100}$

$= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{99}-\frac{1}{100}$

$= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{100}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{100}\right)$

$= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{100}\right)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{50}\right)$

$\Rightarrow A = \frac{1}{51}+\frac{1}{52}+\ldots+\frac{1}{100}$ .....(i)

Now, $B = \frac{1}{51.100}+\frac{1}{52.99}+\ldots+\frac{1}{100.51}$

$= \frac{1}{151}\left(\frac{151}{51.100}+\frac{151}{52.99}+\ldots+\frac{151}{100.51}\right)$

$= \frac{1}{151}\left(\frac{51+100}{51.100}+\frac{52+99}{52.99}+\ldots+\frac{100+51}{100.51}\right)$

$= \frac{1}{151}\left(\frac{1}{100}+\frac{1}{51}+\frac{1}{99}+\frac{1}{52}+\ldots+\frac{1}{51}+\frac{1}{100}\right)$

$\Rightarrow B = \frac{2}{151}\left(\frac{1}{51}+\frac{1}{52}+\ldots+\frac{1}{100}\right)$

$\Rightarrow B = \frac{2}{151}A$ [from equation (i)]

$\frac{A}{B} = \frac{151}{2}$

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8. Let a, b, c be real numbers, not all of them are equal. Prove that if $a + b + c = 0$ then $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2$.

Prove the converse, if $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2$ then $a + b + c = 0$.

Sol. If $a + b + c = 0$, then $(a + b + c)^2 = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = 0$ ....(i)

Also, $a + b = -c$

$\Rightarrow (a + b)^2 = (-c)^2$

$\Rightarrow a^2 + b^2 + 2ab = c^2$ ....(ii)

On adding equation (i) and (ii) we get

$2a^2 + 2b^2 + 2ab = -2(ab + bc + ac)$ ....(iii)

Similarly we get

$2b^2 + 2c^2 + 2bc = -2(ab + bc + ac)$ .... (iv)

$2c^2 + 2a^2 + 2ac = -2(ab + bc + ac)$ .... (v)

therefore from equation (iii), (iv) and (v) we get

$a^2 + b^2 + ab = b^2 + c^2 + bc = c^2 + a^2 + ac$

Converse

If $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ac + a^2$

$a^2 + ab + b^2 = b^2 + bc + c^2$

$\Rightarrow b^2 - a^2 + bc - ac = 0$

$\Rightarrow (b - a)(b + a + c) = 0$ ....(i)

Similarly $(a - c)(b + a + c) = 0$ ....(ii)

and $(c - b)(a + b + c) = 0$ ... (iii)

If $a + b + c \neq 0$ then, from equations (i),(ii) and (iii)

$a - b = 0, b - c = 0, c - a = 0$

$\Rightarrow a = b = c$

but it is given that a, b, c are not equal

therefore $a + b + c = 0$