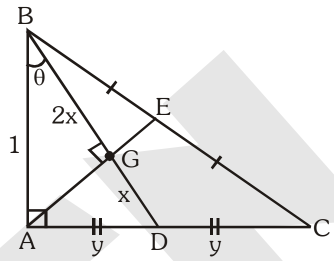
SOLUTION
THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
BHASKARA CONTEST - FINAL - JUNIOR
CLASS - IX & X

Instructions:
1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

1. ABC is a right angled triangle with BC as hypotenuse. The medians drawn to BC and AC are perpendicular to each other. If AB has length 1 cm, find the area of triangle ABC.

Sol. Clearly a is median of ΔABC



In ΔBAG,

$\cos\theta = \frac{2x}{1}$ ....(i)

In ΔABD,

$\cos\theta = \frac{1}{3x}$ ....(ii)

from (i) and (ii), $2x = \frac{1}{3x} \Rightarrow x^2 = \frac{1}{6}$

Now, In ΔABD, $(3x)^2 = y^2 + (1)^2$

$\Rightarrow y^2 = 9x^2 - 1$

$= 9 \times \frac{1}{6} - 1$

$\therefore y^2 = \frac{1}{2} \Rightarrow y = \frac{1}{\sqrt{2}}$

Hence AC $= 2 \times \frac{1}{\sqrt{2}}$

$\therefore [ABC] = \frac{1}{2} \times AB \times AC = \frac{1}{2} \times 2 \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

Hence $[ABC] = \frac{1}{\sqrt{2}}$

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2. (a) Find the smallest positive integer such that it has exactly 100 different positive integer divisors including 1 and the number itself.

(b) A rectangle can be divided into 'n' equal squares. The same rectangle can also be divided into (n + 76) equal squares. Find n.

Sol. (a) Let the number be $N = P_1^{m_1} \cdot P_2^{m_2} \cdot ....P_k^{m_k}$ here P_1, P_2 ..... P_k are the prime divisors.

So, number of divisions = $(m_1 + 1)(m_2 + 1)$ .... $(m_k + 1)$. We need to find smallest N, such that $(m_1 + 1)(m_2 + 1)$ .... $(m_k + 1) = 100$.

Now, lets break 100 into product of decreasing numbers.

i.e., $100 = 100 \times 1, 50 \times 2, 25 \times 4, 20 \times 5, 10 \times 10,$

$25 \times 2 \times 2, 10 \times 5 \times 2, 5 \times 5 \times 4, 5 \times 5 \times 2 \times 2$

Now, for the number to be smallest. It should be

$N = 2^{99}, 2^{49} \times 3, 2^{24} \times 3^3, 2^{19} \times 3^4, 2^9 \times 3^9, 2^{24} \times 3 \times 5, 2^9 \times 3^4 \times 5, 2^4 \times 3^4 \times 5^3, 2^4 \times 3^4 \times 5 \times 7$

Now, amongst all of these numbers

$N = 2^4 \times 3^4 \times 5 \times 7 = 45360$, gives the least value

So, N = 45360

(b) Let 'a' be the side length of squares in 1st condition and 'b' be the side length of square in 2nd condition.

$\therefore na^2 = (n + 76)b^2$ where $n \in I^+$ and $a, b \in R$.

$$\Rightarrow \left(\frac{a}{b}\right)^2 = \frac{n+76}{n} = 1 + \frac{76}{n}$$

Now since, $\left(\frac{a}{b}\right)^2$ is a rational, so we will have two cases

Case 1: If $n | 76$

So n can be 1, 2, 38, 76

But in any case $1 + \frac{76}{n} \neq (k)^2$

Case 2: If $n \nmid 76$

Then n + 76 and n both must be perfect sqaures.

So, let $n + 76 = \ell^2$ and $n = m^2$

So $\ell^2 - m^2 = 76$

$(\ell + m)(\ell - m) = 76$

$= 76 \times 1$

$= 38 \times 2$

$= 19 \times 4$

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$\left.\begin{array}{l}\ell+m=76\\ \ell-m=1\end{array}\right\}$ no integral solution

$\left.\begin{array}{l}\ell+m=38\\ \ell-m=2\end{array}\right\}$ $\ell = 20, m = 18$

$\left.\begin{array}{l}\ell+m=19\\ \ell-m=4\end{array}\right\}$ no integral solution

So, $n = m^2$

$n = (18)^2$

$n = 324$ is the only solution

3. Prove that $1^n + 2^n + 3^n + \ldots + 15^n$ is divisible by 480 for all odd $n \geq 5$.

Sol. $S = 1^n + 2^n + 3^n + \ldots\ldots + 15^n$

Divisibility by 3 :

$S = (1^n + 5^n) + (2^n + 13^n) + (3^n + 6^n) + (4^n + 8^n) + (7^n + 11^n) + (9^n + 12^n) + (10^n + 14^n) + (15^n)$

$\because$ $a^n + b^n$ is divisible by $a + b$, if n is odd. So pairing the number like above, proves that S is divisible by 3.

Divisibility by 5 :

$S = (1^n + 14^n) + (2^n + 13^n) + (3^n + 12^n) + (4^n + 11^n) + (5^n + 10^n) + (6^n + 9^n) + (7^n + 8^n) + (15^n)$

$\because$ $a^n + b^n$ is divisible by $(a + b)$ if n is odd, so pairing the numbers like above, proves that S is divisible by 5.

Divisibility by 32 :

$S = (2^n + 4^n + 6^n + 8^n + 10^n + 12^n + 14^n) + (1^n + 15^n) + (3^n + 13^n) + (5^n + 11^n) + (7^n + 9^n)$

For any odd $n \geq 5$, $2^n + 4^n + \ldots\ldots + 14^n \equiv 0 \pmod{32}$

now, $1^n + 15^n \equiv 16 \pmod{32}$

$3^n + 13^n \equiv 16 \pmod{32}$

$5^n + 11^n \equiv 16 \pmod{32}$

$7^n + 9^n \equiv 16 \pmod{32}$

Adding all, we get

$1^n + 15^n + 3^n + 13^n + 5^n + 11^n + 7^n + 9^n \equiv 64 \pmod{32}$

$\equiv 0 \pmod{32}$

So, S is divisible by $3 \times 5 \times 32 = 480$.

4. Is it possible to have 19 lines in a plane such that (1) no three lines have a common point and (2) they have exactly 95 points of intersection. Validate.

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Sol. Let us take two sets of parallel lines one having 12 lines, another having 5 lines also let us take 2 non parallel lines. {no three lines are concurrent}

Now, total number of points of intersection will be

$= 12 \times 5 + 12 \times 1 + 12 \times 1 + 5 \times 1 + 5 \times 1 + 1 \times 1$

$= 95$

$\therefore$ Yes, it is possible

Altenate :

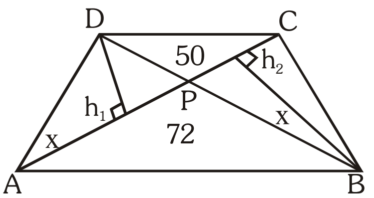
Again let us take two sets of parallel lines, one having 11 lines, one having 7 lines and 1 line not parallel to previous lines.

$\Rightarrow$ Number of points of intersections $= {}^{19}C_2 - {}^{11}C_2 - {}^{7}C_2 = 95$

Which again leads to the conclusion that yes, it is possible.

5. In a trapezium ABCD with AB parallel to CD, the diagonals intersect at P. The area of ΔABP is 72 cm^2 and of ΔCDP is 50 cm^2. Find the area of the trapezium.

Sol.



Let area ΔADP = area $\Delta CBP = x$

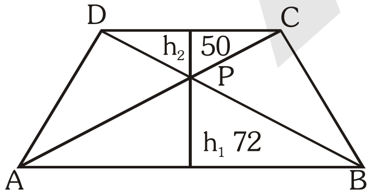
$$\frac{ar\Delta ABP}{ar\Delta ADP} = \frac{BP}{DP} = \frac{ar\Delta BPC}{ar\Delta DPC}$$

$$= \frac{72}{x} = \frac{x}{50} \Rightarrow x = 60$$

ar ABCD $= 2x + 72 + 50$

$= 120 + 72 + 50 = 242$

Alternate



$\Delta ABP \sim \Delta CDP$

$$\frac{ar\Delta ABP}{ar\Delta CDP} = \left(\frac{AB}{CD}\right)^2 \Rightarrow \frac{72}{50} = \left(\frac{AB}{CD}\right)^2$$

$$\frac{36}{25} = \left(\frac{AB}{CD}\right) \Rightarrow \frac{AB}{CD} = \frac{6}{5}$$

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Let AB = 6x
CD = 5x

Then ar $\Delta ABP = \frac{1}{2}$ 6x, $h_1 = 72$

$$\boxed{h_1 = \frac{24}{x}}$$

ar $\Delta CDP = \frac{1}{2} \times 5x \times h_2 = 50$

$$\boxed{h_2 = \frac{20}{x}}$$

$h = h_1 + h_2 = \frac{44}{x}$

area ABCD = $\frac{1}{2}$(AB + CD)h

$= \frac{1}{2}\ (6x + 5\ x) \times \frac{44}{x}$

$= \frac{1}{2} \times 11x \times \frac{44}{x} = 242\ cm^2$

6. Let a < b < c be three positive integers. Prove that among any 2c consecutive positive integers there exists three different numbers x, y, z such that abc divides xyz.

Sol. Given a < b < c
for every 2c consecutive number, there are two number, which are divisible by c.
say r, r + c are divisible by c.
now, since a < b < c
atleast 2a and 2b are in 2c which are divisble by a and b respectively
Say m, m + a divisible by a
n, n + b divisible by b.
let r ≠ m ≠ n
then take r = x, m = y & n = 2
So, xyz is divisible by abc.
If r = m ≠ n
then r + c ≠ m + a ≠ n or n + b
let r + c = x
m + a = y
n or n + b = 2

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Then xyz is again divisible by abc.

Similiarly if $r \neq m = n$

and if $r = m = n$, then

$r + a \neq m + b \neq n + c$

so take $r + a = x$

$m + b = y$

$n + c = 2$

So xyz is again divisible by abc.

7. (a) Let m, n be positive integers. If $m^3 + n^3$ is the square of an integer, then prove that $(m + n)$ is not a product of two different prime numbers.

(b) a,b,c are real numbers such that. $ab + bc + ca = -1$. Prove $a^2 + 5b^2 + 8c^2 \geq 4$

Sol. (a) $m^3 + n^3 = k^2$

$(m + n)(m^2 - mn + n^2) = k^2$

To prove : m + n is not a product of two prime numbers.

let us consider several cases

Case 1: HCF of $(m, n) = 1$

then, let G be the GCD of $(m + n, m^2 + n^2 - mn)$

So $G \mid (m + n)^2 - (m^2 + n^2 - mn)$

$G \mid 3mn$

also$G \mid m + n$

But GCD of $(m + n, mn) = 1$

So $G \mid 3$

Now, if $m + n = pq$, p, q are different primes.

then $pq \mid k^2$

$\Rightarrow pq \mid k$

$\Rightarrow p^2q^2 \mid k^2$

$\Rightarrow pq \mid m^2 - mn + n^2$

so $pq \mid (m^2 + n^2 - mn, m + n)$

$\Rightarrow pq \mid 3$ which gives a contradiction.

Case 2: HCF of $(m, n) = D$ (not equal to 1)

Let $m = Dp$, $n = Dq$, GCD $(p, q) = 1$

$m + n = D(p + q) = p_1p_2$

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So D is prime, p + q is prime which is different from D.

Also $D(p + q) \mid m^2 + n^2 - mn$

So $p + q \mid D^2 (p^2 + q^2 - pq)$

p + q is prime different from D.

But GCD $(p + q, p^2 + q^2 - pq) = 1$ or 3

GCD $(p, q) = 1$

So $p + q = 3$

Now let $p = 1, q = 2$

$p^2 + q^2 - pq = 3$

and equation becomes

$D^3 (3)(3) = k^2$

$k^2 = 9D^3$

Which is contradiction as D is a prime.

(b) $ab + bc + ca = -1$

$a^2 + 5b^2 + 8c^2 - 4 \geq 0$

$a^2 + 5b^2 + 8c^2 + 4(-1) \geq 0$

$a^2 + 5b^2 + 8c^2 + 4(ab + bc + ca) \geq 0$

$a^2 + 5b^2 + 8c^2 + 4ab + 4bc + 4ca \geq 0$

$(a)^2 + (2b)^2 + (2c)^2 + 2(a)(2b) + 2(2b)(2c) + 2(2c)(a) + b^2 + 4c^2 - 4bc \geq 0$

$(a + 2b + 2c)^2 + (b - 2c)^2 \geq 0$

Now since a, b, c are all real numbers, and we shown sum of two perfect squares is greater than or equal to zero. Hence, this completes the proof.

8. ABCD is a quadrilateral in a circle whose diagonals intersect at right angles. Through O the centre of the circle, GOG' and HOH' are drawn parallel to AC, BD respectively, meeting AB, CD in G,H and DC, AB produced in G', H'. Prove GH, G'H' are parallel to BC and AD respectively.

Sol. Given that

$AC \perp BD$ (Diagonal) perpendicular

From the centre 'O' , $OM \perp AC$ and $ON \perp DB$.

as $AC \| GOG^1$ and $BD \| HOH^1$

we know that OM and ON are perpendicular bisectors of AC and BD respectively.

Now $\Delta GND \cong \Delta GNB$ and $\Delta HMA \cong \Delta HMC$ (R.H.S.)

Let $\angle ABD = a = \angle ACD$ (ABCD cyclic, made by same segment AD.)

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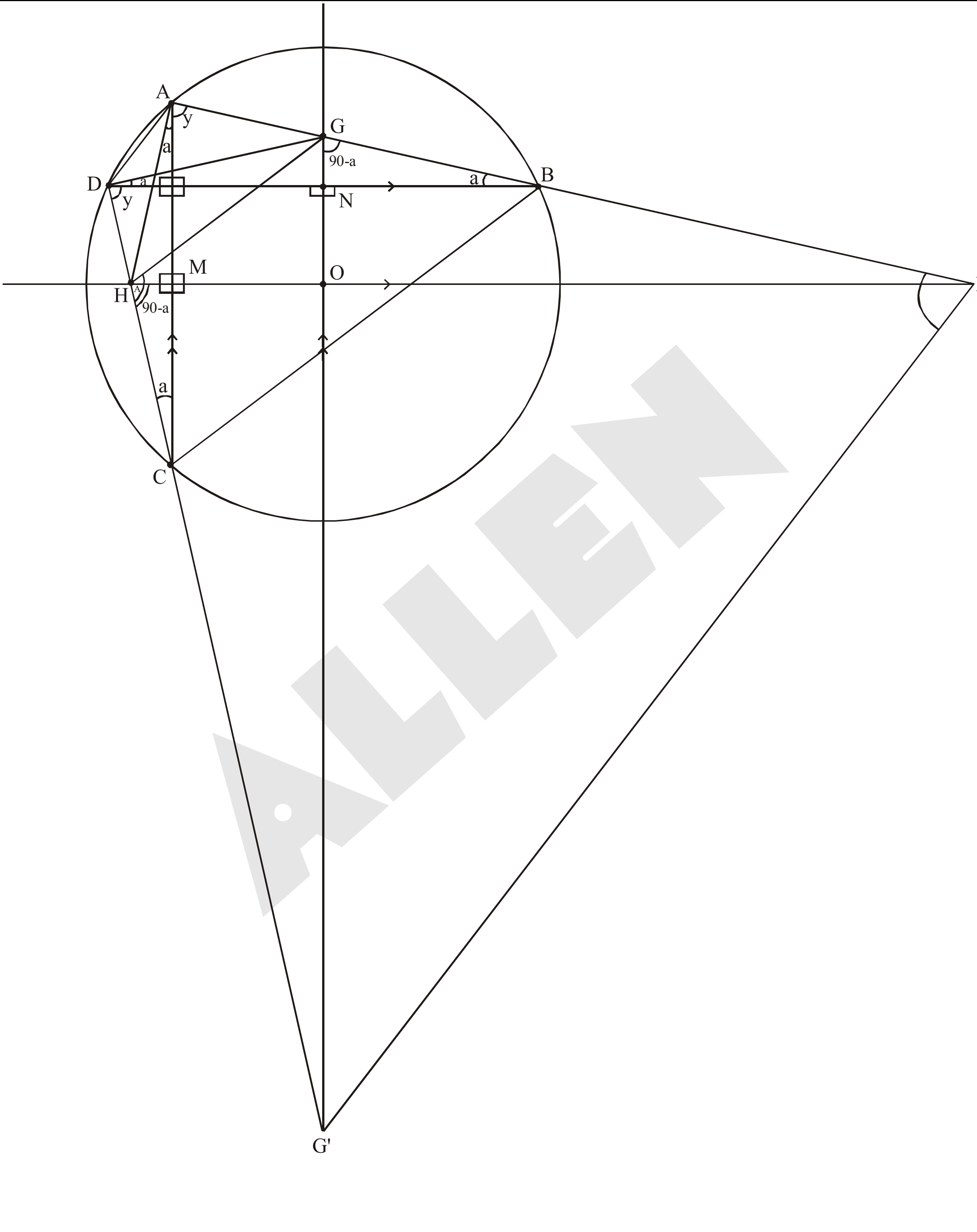

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By congruence triangles we have

$\angle GBN = \angle GDN = a$

$\angle HCM = \angle HAM = a$

Let $\angle CAB = y = \angle CDB$ (Angle made by same segment BC)

$\therefore \angle HAG = a + y = \angle GDH$

$\Rightarrow$ ADHG is cyclic quadrilateral

$\Rightarrow \angle GAD = \angle GHC = \angle A$ (Exterior angle property)

But $\angle DCB = 180 - \angle A$ (As ABCD is cyclic)

$\therefore \angle GHC + \angle BCH = \angle A + 180 - \angle A = 180°$

$\Rightarrow GH||BC$

Now $\angle GBN = a \Rightarrow \angle BGN = 90 - a$

Also $\angle MCH = a \Rightarrow \angle CHM = 90 - a$

$\therefore \angle G'HH' = \angle G'GH' = 90 - a$

$\Rightarrow HGH'G' \rightarrow$ is cyclic quadrilateral

$\Rightarrow \angle GH'G' = \angle GHD = 180 - \angle A$ (Exterior angle property)

$\therefore \angle GH'G' + \angle H'AD = 180 - \angle A + \angle A = 180°$

$\therefore H'G'||AD$