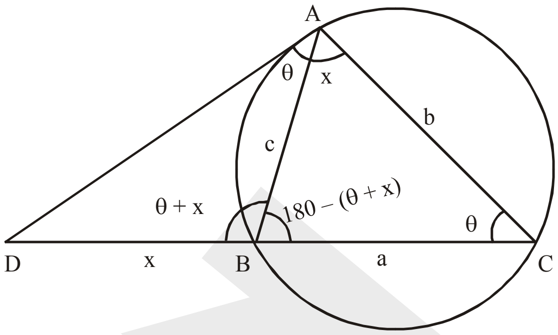
REGIONAL MATHEMATICAL OLYMPIAD (RMO) - 2018

ALLEN CAREER INSTITUTE KOTA (RAJASTHAN) — Path to success

Date: 07/10/2018

Max. Marks: 102 **SOLUTIONS** **Time allowed: 3 hours**

1. Let ABC be a triangle with integer sides in which AB < AC. Let the tangent to the circumcircle of triangle ABC at A intersect the line BC at D. Suppose AD is also an integer. Prove that gcd (AB, AC) > 1.



Sol. Given AB < AC

i.e $c < b$

$\Rightarrow$ $b - c > 0$ .....(1)

Also by tangent-secant theorem

$AD^2 = DB.DC$

$= x (x + a)$

Now we know that $\angle DAB = \angle ACD = \theta$ (Alternate angles)

$\therefore$ $\angle DBA = \theta + x$ Also $\angle ABC = 180 - (\theta + x)$

Now using sine rule in ΔABC $\frac{b}{c} = \sin\frac{(180-(\theta+x))}{\sin\theta} = \frac{\sin(\theta+x)}{\sin\theta}$

Now using sine rule in ΔADB $\frac{AD}{BD} = \frac{\sin(\theta+x)}{\sin\theta} = \frac{b}{c}$

$\therefore$ $AD = BD.\frac{b}{c} = \frac{bx}{c}$ .....(2)

$\therefore$ $AD = \frac{bx}{c} = \frac{ac^2}{b^2-c^2}\times\frac{b}{c} = \frac{abc}{b^2-c^2}$

Now $AD^2 = x(x + a) = x^2 \frac{b^2}{c^2}$

$\Rightarrow$ $x\left(\frac{b^2}{c^2}-1\right) = a$

$\Rightarrow$ $x = \frac{ac^2}{b^2-c^2}$

Suppose gcd(b, c) = 1

Now $AD = \frac{abc}{b^2-c^2} \Rightarrow b^2 - c^2 \mid a$

$\Rightarrow$ $a > b^2 - c^2$

$a > (b + c)(b - c)$

We have $b - c > 0$ from (1)

$\Rightarrow \; a > b + c$

But by triangle inequality we have $b + c > a$

we got contradiction by our assumption

$\therefore \; \gcd(b, c) > 1$

2. Let n be a natural number. Find all real numbers x satisfying the equation

$$\sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}$$

Sol. Claim : $\dfrac{x^k}{1+x^{2k}} \le \dfrac{1}{2}$

$\Leftrightarrow \; 2x^k \le 1 + x^{2k}$

$\Leftrightarrow \; (1 - x^k) \ge 0$

equality for $x = 1$ (for any $k \in N$)

Now $\dfrac{k\,x^k}{1+x^{2k}} \le \dfrac{k}{2}$

$$\Rightarrow \; \sum_{k=1}^{n} \frac{k\,x^k}{1+x^{2k}} \le \sum_{k=1}^{h} \frac{k}{2} = \frac{n(n+1)}{4}$$

Hence equality must hold, which will hold for $x = 1$.

Alternate solution

For $n \in N$, and we need to find all $x \in R$ such that

$$\sum_{k=1}^{n} \frac{k\,x^k}{1+x^{2k}} = \frac{n(n+1)}{4}$$

$$\Rightarrow \; \sum_{k=1}^{n} \frac{2kx^k}{1+x^{2k}} = \frac{n(n+1)}{2} = \sum_{k=1}^{n} k \quad \left(\because \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \right)$$

$$\Rightarrow \; \sum_{k=1}^{n} \left[\frac{2k\,x^k}{1+x^{2k}} - k \right] = 0$$

$$\Rightarrow \; \sum_{k=1}^{n} k \left[\frac{2x^k - 1 - x^{2k}}{1+x^{2k}} \right] = 0$$

$$\Rightarrow \; k.\sum_{k=1}^{n} \left[\frac{\left(x^k - 1\right)^2}{1+x^{2k}} \right] = 0$$

As $k > 0 \Rightarrow (x^k - 1)^2 = 0$

$\Rightarrow x^k = 1$

$\Rightarrow x = 1$ is only Real solution.

3. For a rational number r, its period is the length of the smallest repeating block in its decimal expansion. For example, the number r = 0.123123123.....has period 3. If S denotes the set of all rational number r of the form r = 0.abcdefgh having period 8, find the sun of all the elements of S.

Sol. $x = 0.\overline{abcdefgh}$

10^8 x = abcdefgh . $\overline{abcdefgh}$

(10^8-1) x = abcdefgh

$$x = \frac{abcdefgh}{10^8 - 1}$$

So, abcdefgh canbe 1, 2, 3, ..., $10^8 - 2$

$$\Rightarrow \ \Sigma x = \frac{1}{10^8 - 1} \sum_{r=1}^{10^8-2} r$$

$$= \frac{1}{10^8-1} \frac{(10^8-2)(10^8-1)}{2}$$

$= 5 \times 10^7 - 1$

In this we have added period 1, period 2 and period 4 numbers also.

Let as find sum of period 1 or 2 or 4 = y (say)

$$\Sigma y = \frac{1}{10^4-1} \sum_{r=1}^{10^4-2} r = \frac{1}{(10^4-1)} \frac{(10^4-2)(10^4-1)}{2}$$

$= 5 \times 10^3 - 1$

$\Rightarrow$ Req answer $= (5 \times 10^7 - 1) - (5 \times 10^3 - 1)$

$= 5 \times 10^3 (10^4 - 1)$

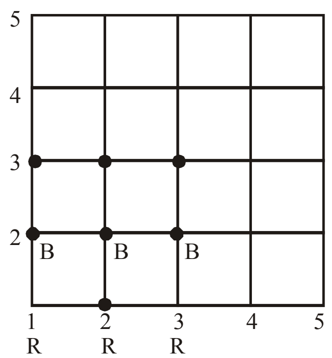
$= 49995000$

4. Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \le m \le 5$, $1 \le n \le 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form (a, b), (a + k, b), (a + k, b + k), (a, b + k), for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square with sides parallel to axes and having at least three points of the same colour.)

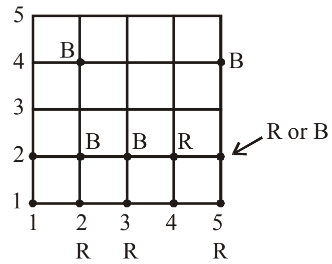
Sol. As there are five points on y = 1

By PHP (pigeon hole principle), at least 3 must be of same colour say 'Red'.

There can be following cases :



(1) All three consecutive points have same colour, W.L.O.G. say (1, 1) (2, 1) (3, 1) are red, then out of (1, 2), (2, 2), (3, 2) any of red colour we are done! Let us assume all are blue. Now out of (1, 3), (2, 3), (3, 3) any blue then we are done! and if not then all will be red and we will get a 2×2 square with all red corner!!



Case (2) Two adjacent points and one not adjacent point have same colour

Let (2, 1), (3, 1) and (5, 1) are red then (2, 2), (3, 2), (2, 4), (5, 4)

Must be blue, other wise we are done!

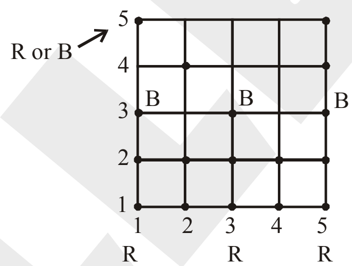
Now (4, 2) must be red. If (5, 2) Red we are done as (4, 2), (5, 1), (5, 2) are red

If it is blue then also as (3, 2), (5, 2), (5, 4) are blue.

Similarly we can do it the case (1, 1), (2, 1), (5, 1) are red

Case (3) No adjacent points have same colour

Let (1, 1), (3, 1), (5, 1) are red



then (1, 3), (3, 3), (5, 3) must be blue now at (1, 5) whether we paint Red or blue we are done!!

5. Find all natural numbers n such that $1 + [\sqrt{2n}]$ divides 2n. (For any real number x, [x] denotes the largest integer not exceeding x.)

Sol. Let $m \le \sqrt{2m} < m + 1$ for all $m \in N$

$m^2 \le 2n < (m + 1)^2$

Now, $1 + \sqrt{2n} = m + 1$

Now we have to find all natural number m, n such that m + 1 should divide 2n

So, m + 1 divides $m^2 - 1$ and next numbers

After $m^2 - 1$ are $m^2 - 1 + m + 1 = m^2 + m$

$m^2 - 1 + 2(m+1) = (m + 1)^2$

$\therefore$ A1so $2n < (m + 1)^2$ and $m^2 - 1 < 2n$.

From above $\quad 2n = m\ (m + 1)$

$$n = \frac{m(m+1)}{2}$$

As $m \in N$, all the number in the form of $\frac{m(m+1)}{2}$ are called as triangular numbers

So our solution is all triangular numbers $m \in N$

Such as 1, 3, 6, 10 ....

Verification

$n = 3 \quad 1 + \left[\sqrt{2n}\right] = 1 + \left[\sqrt{2 \times 3}\right] = 1 + \left[\sqrt{6}\right] = 1 + 2 = 3$

$2n = 6$ $\qquad$ 3 divides 6

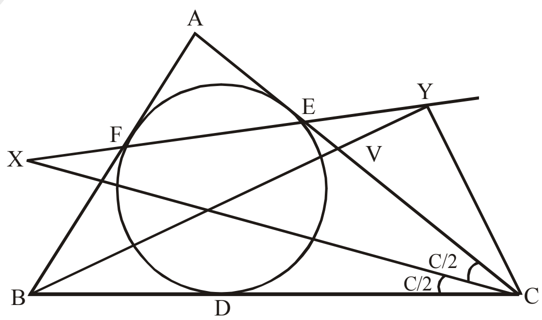
$n = 10 \quad 1 + \left[\sqrt{2 \times 10}\right] = 1 + 4 = 5$ $\qquad$ 5 divides 20

$2n = 20$ $\qquad$ so .... on

6. Let ABC be an acute-angled triangle with AB < AC. Let I be the incertre of triangle ABC, and let D, E, F be the points at which its incircle touches the sides BC, CA, AB, respectively. Let BI, CI meet the line EF at Y, X, respectively. Further assume that both X and Y are outside the triangle ABC. Prove that

(i) B, C, Y, X are concyclic; and (ii) I is also the incentre of triangle DYX.

Sol. (i) $\angle AFE = \angle AEB = 90 - \frac{A}{2}$



$$\angle BVC = 180 - \left(\frac{B}{2} + C\right) = \angle EVY$$

$$\angle EYV = 180 - \left(90 - \frac{A}{2}\right) - \left[180 - \left(\frac{B}{2} + C\right)\right]$$

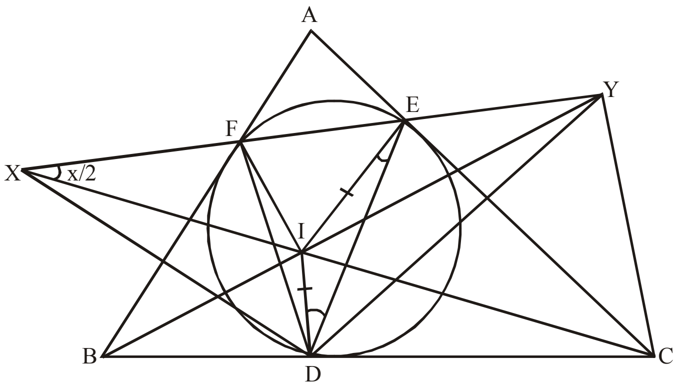
$$= 180 - 90 + \frac{A}{2} - 180 + \frac{B}{2} + C$$

$$= -\frac{A}{2} - \frac{B}{2} - \frac{C}{2} + \frac{A}{2} + \frac{B}{2} + C$$

$$\angle BYX = \frac{C}{2} = \angle BCX$$

$\therefore$ BCYX are concyclic

(ii) Now $\angle EDC = 90 - \frac{c}{2}$



$\angle IDE = 90 - \left(90 - \frac{c}{2}\right) = \frac{c}{2}$

$\angle IDE = \angle IYE = \frac{c}{2}$

IEYD a is cyclic quadrilateral

also, ID = IE

$\angle IED = \angle IDE = \frac{c}{2}$

$\angle IED = \angle IYD = \frac{c}{2} = \angle IYE$

$\therefore$ IF bisects $\angle EYD$

Similarly,

$\angle BDF = \angle BFD = 90 - \frac{B}{2}$

$\angle IDF = 90 - \left(90 - \frac{B}{2}\right) = \frac{B}{2} = \angle IXF$

So IDXF is also concyclic quadrilateral

ID = IF

$\angle IDF = \angle IFD = \frac{B}{2}$

$\angle IFD = \angle IXD = \frac{B}{2}$

Thus, IX bisects $\angle DXY$

Similarly we can prove IY bisects $\angle DYX$

$\therefore$ I is the incentre of ΔDXY