PRE REGIONAL MATHEMATICAL OLYMPIAD (PRMO) - 2018

Date: 19/08/2018

Max. Marks: 102 **SOLUTIONS** **Time allowed: 3 hours**

1. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n?

Ans. (17)

Sol. Let the number of pages in the first book be x

$x, x + 50, \frac{3}{2}(x + 50)$

$1 + (x + 1) + (x + x + 50 + 1) = 1709$

$3x + 53 = 1709$

$3x = 1656$

$x = 552$

Last page number $= x + x + 50 + \frac{3}{2}(x + 50)$

$= 552 + 552 + 50 + \frac{3}{2}(552 + 50)$

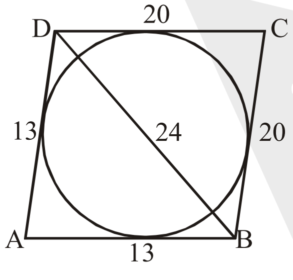
$= 2057$

Largest prime factor of 2057 = 17

2. In a quadrilateral ABCD, it is given that AB = AD = 13, BC = CD = 20, BD = 24. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

Ans. (08)

Sol.



[ABD] = 60, [DBC] = 192, [ABCD] = 252

$$\frac{13r}{2}+\frac{13r}{2}+\frac{20r}{2}+\frac{20r}{2} = 252$$

$33r = 252$

$r = \frac{252}{33} = 7.63$ $\therefore$ Nearest integer = 8

3. Consider all 6-digit numbers of the form abccba where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.

Ans. (70)

Sol. abccba is divisible by 7

if abc – cba is divisible by 7

$\Rightarrow$ abc – cba = 99 (a – c) = 7 M

So, (a, c) = {(9, 2), (8, 1), (7, 0), (2, 9), (1, 8), (9, 9), (8, 8), (7, 7), (6, 6), (5, 5), (4, 4), (3, 3), (2, 2), (1, 1)}

No of pair of (a, b) = 14

Also, no of b's can be = 5

$\therefore$ Total number of 6 digits number = $(14 \times 5) = 70$

4. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is, 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.

Ans. (12)

Sol. $166 \times 56 = 8590$

$(n^2 + 6n + 6) \times (5n + 6) = (8n^3 + 5n^2 + 9n + 0)$

$5n^3 + 30n^2 + 30n + 6n^2 + 36n + 36 = 8n^3 + 5n^2 + 9n$

$f(n) = 3n^3 - 31n^2 - 57n - 36 = 0$

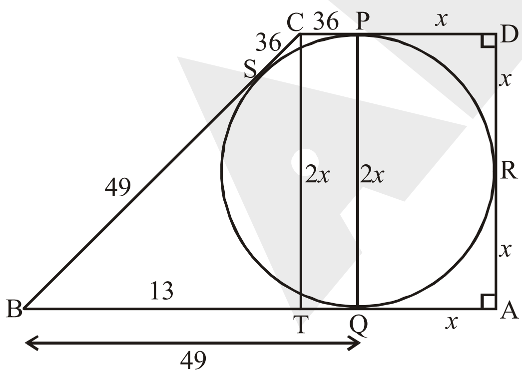
at $n = 12$ $\qquad f(n) = 0$

so base $n = 12$

5. Let ABCD be a trapezium in which AB || CD and AD $\perp$ AB. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49, find PQ.

Ans. (84)

Sol.



$CP = TQ = 36 \Rightarrow BT = 49 - 36 = 13$

In $\Delta BTC \Rightarrow 85^2 = 13^2 + (2x)^2$

$\Rightarrow (2x)^2 = 7056 \Rightarrow 2x = 84$

$\therefore$ PQ = 84 cm

6. Integers a, b, c satisfy $a + b - c = 1$ and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

Ans. (18)

Sol. $a^2 + b^2 = c^2 - 1$

$a + b = c + 1$

$a^2 + b^2 + 2ab = c^2 + 1 + 2c$

$c - 1 + 2ab = c^2 + 1 + 2c$

$2ab = 2c + 2$

$ab = c + 1$

$a + b = ab$

$\frac{1}{a}+\frac{1}{b} = 1$

$a = b$

$a = b = 0,\ c = -1$

$a = b = 2,\ c = 3$

$8 + 9 = 17$

$17 + 1 = 18$

$(a - 1)(b - 1) = 1$

$a - 1 = 1,\ b - 1 = 1,\ a = b = 2$

$a - 1 = -1,\ b - 1 = -1,\ a = b = 0$

7. A point P in the interior of a regular hexagon is at distance 8, 8, 16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r?

Ans. (14)

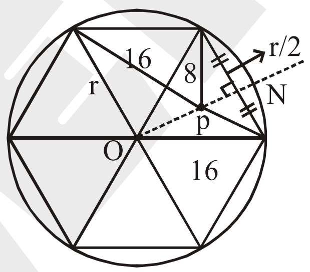
Sol. $PN = \sqrt{64-\frac{r^2}{4}} \Rightarrow OP = \frac{\sqrt{3}r}{2} - PN$

$\Rightarrow \sqrt{256-r^2} = \frac{\sqrt{3}r}{2}-\sqrt{64-\frac{r^2}{4}}$

$\Rightarrow r^4 - 256 r^2 + 12288 = 0$

$\Rightarrow r^2 = 64,\ 192 \Rightarrow r = \sqrt{192}$

$\Rightarrow 14$



8. Let AB be a chord of a circle with centre O. Let C be a point on the circle such that $\angle ABC = 30°$ and O lies inside the triangle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20°$. Find the measure of $\angle CDO$ in degrees.

Ans. (80)

Sol. $\angle ABC = 30°$

$\therefore\ \angle AOC = 60°$ with $OC = OA$, so

ΔOAC is equilateral $OA = OC = AC = R$

Join OB, since $OC = OB$ so

$\angle OBC = \angle OCB = 20°$

$\angle OBA = 10°,\ \angle OAB = 10°,\ \angle OAC = 60°,\ \angle DAC = 70°$

Also, $\angle ACO = 60°$

so $\angle ACD = 60° - 20° = 40°$

In ΔACD by ASP

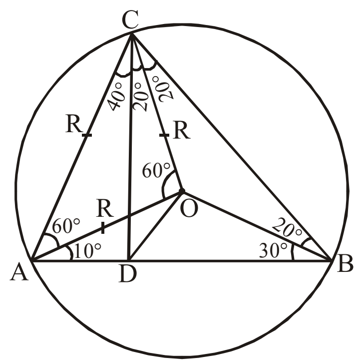
$\angle ADC = 70°$

$\Rightarrow \therefore\ \angle ADC = \angle CAD = 70°$

$AC = CD = R$, In ΔCDO, $CD = CO$

and $\angle DCO = 20°$,

$\therefore\ \angle CDO = \frac{180°-20°}{2} = 80°$



9. Suppose a, b are integers and a + b is a root of $x^2 + ax + b = 0$. What is the maximum possible values of b^2?

Ans. (81)

Sol. $(a + b)^2 + a(a + b) + b = 0$

$\Rightarrow 2a^2 + b^2 + 3ab + b = 0$

$\Rightarrow 2a^2 + a(3b) + b^2 + b = 0$

$\Rightarrow a = \dfrac{3b \pm \sqrt{9b^2 - 8(b^2 + b)}}{4}$

So, $b^2 - 8b$ must be a perfect square

$\Rightarrow (b - 4)^2 - 16 = K^2$; $K \in Z$

$\Rightarrow (b - 4)^2 - K^2 = 16$

$\Rightarrow (b - 4 - K)(b - 4 + K) = 16$

Possible values of (b, K) are

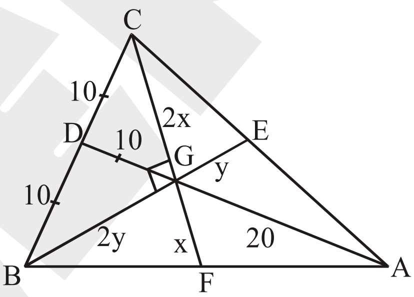
(9, 3), (–1, 3) (4, 0), (0, 0)

Hence, maximum possible value of b^2 is 81

10. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

Ans. (24)

Sol. CD = BD = GD ($\because$ right triangle)



$AB^2 = (2BF)^2 = 4(x^2 + 4y^2)$

$AC^2 = (2CE)^2 = 4(y^2 + 4x^2)$

$BC^2 = 20^2 = 4(x^2 + y^2)$

$AB^2 + BC^2 + AC^2 = 6 \times 4(x^2 + y^2)$

$= 6 \times 20^2 = 2400$

$= \dfrac{2400}{100} = 24$

$x^2 + 4y^2 = \dfrac{b^2}{4}$ $\qquad\qquad$ $\dfrac{6a^2}{100} = \dfrac{a^2 + b^2 + c^2}{100}$

$5(x^2 + y^2) = \dfrac{b^2 + c^2}{4}$

$5(4x^2 + 4y^2) = b^2 + c^2$

Appoloneons theorem,

$b^2 + c^2 = 2(20)^2 + \dfrac{a^2}{2}$

$5a^2 = 2(30)^2 + \dfrac{a^2}{2}$

$5a^2 - \dfrac{a^2}{2} = 3 \times 900$

$\dfrac{a^2}{2} = 3 \times 900$

$\dfrac{a^2}{100} = 4$

$\dfrac{6a^2}{100} = 24$

11. There are several tea cups in the kitchen, some with handles and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?

Ans. (29)

Sol. Let the no. of cups with handles be x and no. of cups without handle be y

$^xC_2 . {^yC_3} = 1200$

$\Rightarrow \dfrac{x!}{(x-2)!2!} \times \dfrac{y!}{(y-3)!3!} = 1200$

$\Rightarrow \dfrac{x(x-1)}{2} \times \dfrac{y(y-1)(y-2)}{6} = 1200$

$\Rightarrow x(x-1) \times y(y-1)(y-2) = 14400$

$\Rightarrow 4 \times 5 \times 8 \times 9 \times 10 = 14400$

$\Rightarrow 24 \times 25 \times 2 \times 3 \times 4 = 14400$

$\therefore$ x = 5 ; y = 10

or x = 25 ; y = 4

$\therefore$ Maximum number of cups = 25 + 4 = 29

12. Determine the number of 8-tuples $(\in_1, \in_2, ..., \in_8)$ such that $\in_1, \in_2, ..., \in_8 \in \{1, -1\}$ and $\in_1 + 2\in_2 + 3\in_3 + ... + 8\in_8$ is a multiple of 3.

Ans. (88)

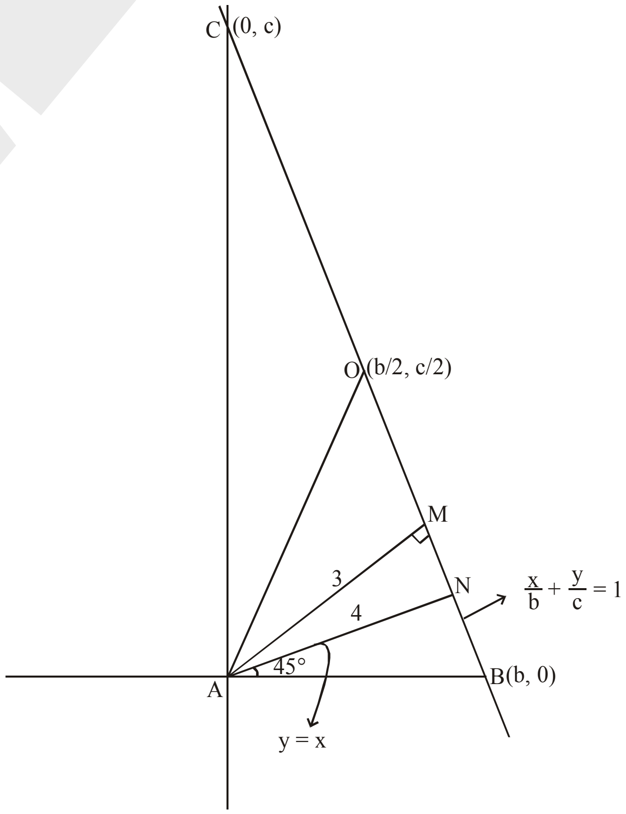
Sol. $\sum_{k=1}^{8} k \in_k \equiv \sum_{k=0}^{2} \in_{3k+1} - \sum_{k=0}^{2} \in_{3k+2} \equiv 0 \pmod 3$

$\Rightarrow \in_1 + \in_4 + \in_7 \equiv \in_2 + \in_5 + \in_8 \pmod 3$ its easy to count now, there 22 Possible $(\in_1, \in_2, \in_4, \in_5, \in_7, \in_8)$ which satisfy thisx where $\in_i \in \{-1, 1\}$.

Thus, there are a total of $4 \times 22 = 88$ solutions, considering $\in_3$ and $\in_6$.

13. In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle$A have lengths 3 and 4, respectively. Find the length of the medium through A.

Ans. (24)



Sol. $N \equiv \left(\dfrac{bc}{b+c}, \dfrac{bc}{b+c}\right)$

So, AM = $\sqrt{2}\dfrac{bc}{b+c} = 4$ .....(i)

$\dfrac{1}{\sqrt{\dfrac{1}{b^2}+\dfrac{1}{c^2}}} = 3$

$\Rightarrow \dfrac{bc}{\sqrt{b^2+c^2}} = 3 = AM$ ... (ii)

$\Rightarrow (bc)^2 = 9[(b+c)^2 - 2bc]$

$\Rightarrow b^2c^2 = 9\left[\dfrac{b^2c^2}{8} - 2bc\right]$

From (i) and (ii)

$\Rightarrow bc = 144 \Rightarrow \sqrt{b^2+c^2} = 48$

Hence, median = $\dfrac{\sqrt{b^2+c^2}}{2} = 24$

14. If x = cos1° cos2° cos3° ... cos89° and y = cos2° cos6° cos10° .... cos86°, then what is the integer nearest to $\frac{2}{7}\log_2(y/x)$?

Ans. (19)

Sol. x = cos1° cos2° cos3° ... cos89°

= (cos1° cos89°) (cos2° cos88°) ... (cos44° cos46°) cos45°

= (cos1° sin1°) (cos2° sin2°) ... (cos44° sin44°) cos45°

$x = \frac{1}{2^{44}}\frac{1}{\sqrt{2}}$ (sin2° sin4° ... sin88°)

$= \frac{1}{2^{44}}\frac{1}{\sqrt{2}}$ (sin2° sin88°) (sin4° sin86°) ... (sin44° sin46°)

$= \frac{1}{2^{44}}\frac{1}{\sqrt{2}}$ (sin2° cos2°) (sin4° cos4°) ... (sin44° cos44°)

$= \frac{1}{2^{66}}\frac{1}{\sqrt{2}}$ (sin4° sin8° ... sin84° sin88°)

$$\frac{y}{x} = \frac{\cos 2°\cos 6°...\cos 82°\cos 86°}{\frac{1}{2^{66}}\frac{1}{\sqrt{2}}(\sin 4° \sin 8°... \sin 84°\sin 88°)} = 2^{\frac{133}{2}}$$ (all terms cancels)

$\Rightarrow \frac{2}{7}\log_2 2^{(133/2)}$

$\Rightarrow \frac{2}{7}\times\frac{133}{2}=\frac{133}{7}=19$

= 19 Ans.

15. Let a and b natural number such that 2a – b, a – 2b and a + b are all distinct squares. What is the smallest possible value of b?

Ans. (21)

Sol. Let 2a – b, a – 2b and a + b be equal to sqaure of natural number n, k and r respectively.

Therefore, $2a - b = k^2$ ....(i)

$a - 2b = n^2$ ....(ii)

and $a + b = r^2$ ....(iii)

(i) - (ii) gives, $a + b = n^2$ ....(iv)

By (iv) and (iii), $k^2 - n^2 = r^2 \Rightarrow k^2 = n^2 + r^2$

Since, $b = \frac{\left(r^2 - \frac{(n^2+k^2)}{3}\right)}{2}$

Also, $n^2 + r^2$ is divisible by 3 so, (n, k, r) = (9, 15, 12)

Minimum value of b is $\frac{\left(12^2 - \frac{(15^2+9^2)}{3}\right)}{2} = 21$

May be smallest b is 21.

16. What is the value of

$$\sum_{\substack{1\le i<j\le 10\\ i+j=odd}} (i+j) - \sum_{\substack{1\le i<j\le 10\\ i+j=even}} (i+j)?$$

Ans. (55)

Sol. $S = \sum_{\substack{1\le i<j\le 10\\ i+j=odd}} (i+j) - \sum_{\substack{1\le i<j\le 10\\ i+j=even}} (i+j)?$

$\Rightarrow S = \sum_{1\le i<j\le 10} (-1)^{i+j-1}(i+j)$

$\Rightarrow S = \frac{1}{2}\sum_{1\le i<j\le 10} (-1)^{i+j-1}(i+j+22-(i+j))$

$\Rightarrow S = \frac{1}{2}\sum_{1\le i<j\le 10} (-1)^{i+j-1}$

Let us count how many times i + j is even or odd,

i + j = even , $2.{}^5C_2 = 20$

i + j = odd , ${}^5C_1.{}^5C_1 = 25$

$\Rightarrow S = 11\ (-20 + 25) = 55$

Aliternate

In even cases we have to exclude 1 = 1 2 = 2 .... 10 = 10 cases as ($i \ne j$)

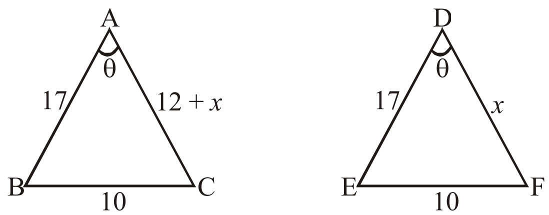
So sum $1 + 2 + ... + 10 = \frac{10\times 11}{2} = 55$

will be difference

17. Triangles ABC and DEF are such that $\angle A = \angle D$, AB = DE = 17, BC = EF = 10 and AC – DF = 12. What is AC + DF?

Ans. (30)

Sol. $\cos\theta = \frac{(17)^2+(12+x)^2-(10)^2}{2(17)(12+x)} = \frac{(17)^2+x^2-(10)^2}{2(17)(x)}$



$= \frac{(289)+(12+x)^2-100}{(12+x)} = \frac{(289)+x^2-100}{x}$

$= \frac{289}{12+x} + (12 + x) - \frac{100}{(12+x)} = \frac{289}{x} + x - \frac{100}{x}$

$= 12 + \frac{189}{12+x} = \frac{189}{x}$

$12 = (189)\left(\frac{1}{x} - \frac{1}{12+x}\right)$

$12 = (189)\left(\frac{12}{x(12+x)}\right)$

$x^2 + 12x - 189 = 0$

$x^2 + 21x - 9x - 189 = 0$

$(x + 21)(x - 9) = 0$

$x = 9$

$\therefore$ AC + DF = 12 + 2x = 12 + 2(9) = 30

18. If a, b, c ≥ 4 are integers, not all equal and 4abc = (a + 3) (b +3) (c + 3), then what is the value of a + b + c?

Ans. **(16)**

Sol. By observation a = 4, b = 5, c = 7

satisfies given equations

$\therefore$ a + b + c = 4 + 5 + 7 = 16

19. Let N = 6 + 66 + 666 +...+ 666 ... 66, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N?

Ans. **(33)**

Sol. $N = \frac{6}{9}[9 + 99 + ... + 999...9]$

$= \frac{6}{9}\left[\frac{10(10^n - 1)}{9} - 100\right]$

$= \frac{6}{9}\left[\frac{10}{9} \times \frac{999...9}{100 \text{ times}} - 100\right]$

$\frac{60}{9}\left[\underbrace{111....1}_{100 \text{ times}} - 10\right]$

$\frac{20}{3}$ [111....101]

$\frac{[222....22020]}{3}$

22020 $\rightarrow$ 7340

22222020 $\rightarrow$ 740730

22222222020 $\rightarrow$ 7407407340

each pair of 222 gives 740 & 22020 gives 7340 in last

So that 33 times occur.

20. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.

Ans. **(17)**

Sol. Let product of digits of n be P(x)

claim : P(n) ≤ n

Proof: Let $n = a_m 10^m + a_m\ 10^{m-1} + ... + a_0$

$\geq a_m 10^m$

$\geq a_m a^m$

$\geq a_m a_{m-1} \ldots a_0$

$= P(n)$

Now $n^2 - 15n - 27 \leq n$

$\Rightarrow n^2 - 16n - 27 \leq 0$

$-1 < 8 - \sqrt{91} \leq n \leq 8 + \sqrt{91} < 18$ ...(i)

Also P(n) ≥ 0

$\Rightarrow\ n^2 - 15n - 27 \geq 0$

$\Rightarrow\ n^2 - 15n + 56 \geq 83$

$\Rightarrow\ (n - 7)(n - 8) \geq 83$

$\Rightarrow\ (n - 7)^2 > (n - 7)(n - 8) \geq 83$

$\Rightarrow\ (n - 7) > \sqrt{83}$

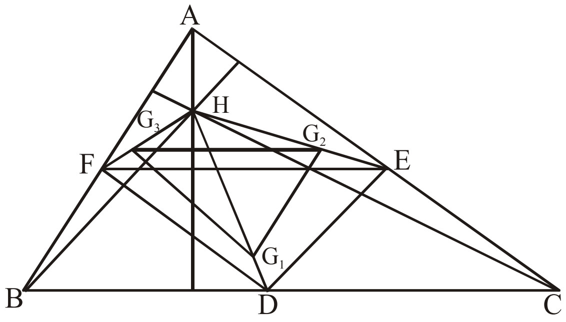
$\Rightarrow\ n > 7 + \sqrt{83} > 16$ ...(ii)

From (i) and (ii) n = 17 which satisfies the condition.

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2 and G_3 be the centroids of the triangles HBC, HCA and HAB, respectively. If the area of triangle $G_1G_2G_3$ is 7 units, what is the area of triangle ABC?

Ans. (63)

Sol.



Given

$$\frac{HG_1}{HD} = \frac{HG_2}{HE} = \frac{2}{3} \text{ and } \frac{HG_1}{HD} = \frac{HG_3}{HF} = \frac{2}{3}$$

$G_1G_2 \parallel DE$ and $G_1G_2 \parallel DF$

and $G_2G_3 \parallel FE$

$$\frac{HG_2}{HE} = \frac{HG_3}{HF} = \frac{2}{3}$$

$G_2G_3 \parallel EF$

$[HG_1G_2] = \frac{4}{9}[HDE]$ (1)

$[HG_1G_3] = \frac{4}{9}[HFD]$ (2)

$[HG_2G_3] = \frac{4}{9}[HFE]$ ... (3)

(1) + (2) –(3)

$[HG_1G_2] + [HG_1G_3] + [HG_2G_3] = \frac{4}{9}([HDE] + [HFD] + [HFE])$

$[G_1G_2G_3] = \frac{4}{9}[DEF]$

$9[G_1G_2G_3] = 4[DEF] = ABC$

$9 \times 7 = [ABC]$

$63 = [ABC]$

22. A positive integer k is said to be good if there exists a partition of {1, 2, 3, ... 20} in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there?

Ans. (06)

Sol. Let us partition it in k part and each part to 's' then,

ks = 1 + 20 + ... + 20

$\Rightarrow$ ks = 210

$\Rightarrow$ s | 210

Also s must be $\geq$ 20, (as 20 will be present in some partition)

Now, 210 = 2 × 3 × 5 × 7

So, Proper divisors of 210 are 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105

$\Rightarrow$ s can be 21, 20, 35, 42, 70, 105

for s = 21, we have

(1, 20), (2, 19), ... (10, 11)

$\Rightarrow$ 21 is good number

for s = 42, join two two pairs of above

for s = 105, join five five pairs of above

$\Rightarrow$ 42 and 105 are also good number.

for s = 70

{5, 6, 7, 8, 9, 10, 11, 14}, {15, 16, 17, 18, 4}, {1, 2, 3, 12, 13, 19, 20}

$\Rightarrow$ s = 70 is a good number

for s = 35

{5, 9, 11, 10} {6, 7, 8, 14}, {4, 15, 16}, {17, 18}, {2, 13, 20}, {1, 3, 12, 19}

s = 35 is also a good number.

For s = 30

{20, 10}, {19, 11}, {18, 12}, {17, 13}, {16, 14}, {15, 9, 6}, {1, 2, 3, 4, 5, 7, 8}

$\Rightarrow$ s = 30 is also a good number

Hence, there are total 6 good number.

23. What is the largest positive integer n such that $\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n(a+b+c)$ holds for all positive real numbers a, b, c

Ans. (14)

Sol. If a, b, c, x, y, z are positive reals,

then, $\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}$

(By extended form of cauchy schwarz),

$$\therefore \frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq \frac{(a+b+c)^2}{\frac{b}{29}+\frac{c}{31}+\frac{c}{29}+\frac{a}{31}+\frac{a}{29}+\frac{b}{31}} \geq \frac{(a+b+c)^2}{\frac{a+b+c}{29}+\frac{a+b+c}{31}}$$

$$\geq \frac{(a+b+c)^2}{(a+b+c)\left[\frac{1}{29}+\frac{1}{31}\right]} \geq \frac{(a+b+c)}{\frac{60}{29.31}} \geq \frac{a+b+c}{\frac{60}{899}}$$

$\geq (a+b+c)\left(\frac{899}{60}\right)$

$\geq$ 14.98 (a + b + c)

$\geq$ n (a + b + c)

$\therefore$ Largest positive integer will be n = 14

24. If N is the number of triangles of different shapes (i.e. not similar) whose angle are all integers (in degreees), what is N/100?

Ans. **(27)**

Sol. Let the angles be $\lambda_1, \lambda_2, \lambda_3$

$\lambda_1 + \lambda_2 + \lambda_3 = 180°$

number of positive solution are

$= {}^{180-1}C_{3-1} = {}^{179}C_2$

But some solutions are counted more than once like,

$$\left.\begin{matrix} 1 & 1 & 178 \\ 1 & 2 & 176 \\ 3 & 3 & 174 \\ \vdots & & \\ 59 & 59 & 62 \\ 61 & 61 & 58 \\ 62 & 62 & 56 \\ & \vdots & \\ 89 & 89 & 2 \end{matrix}\right\} \rightarrow \text{are counted 3 times}$$

$\rightarrow$ 88 solution

and all solution with $\lambda_1 \neq \lambda_2 \neq \lambda_6$ are counted 6 times.

60 60 60 one times

$$\underbrace{\frac{1}{6}\left({}^{179}C_2 - 3\times 88 - 1\right)}_{\text{scalene and all different}} + \underbrace{88}_{\text{isosceles but not equilateral}} + \underbrace{1}_{\text{equilateral}} = N$$

$N = 2700 \Rightarrow \dfrac{N}{100} = 27$

25. Let T be the smallest positive integer which, when divided by 11, 13, 15 leaves remainders in the sets {7, 8, 9}, {1, 2, 3}, {4, 5, 6} respectively. What is the sum of the squares of the digits of T?

Ans. **(81)**

Sol. x = 7, 8, 9 (mode 11)

x = 1, 2, 3 (mode 13)

x = 4, 5, 6 (mode 15)

L.C.M. of 15, 13 is 195

So by observation,

183, 184, 185 = 1, 2, 3 (mode 13)

184, 185, 186 = 4, 5, 6 (mode 15)

183, 184, 185 = 7, 8, 9 (mode 11)

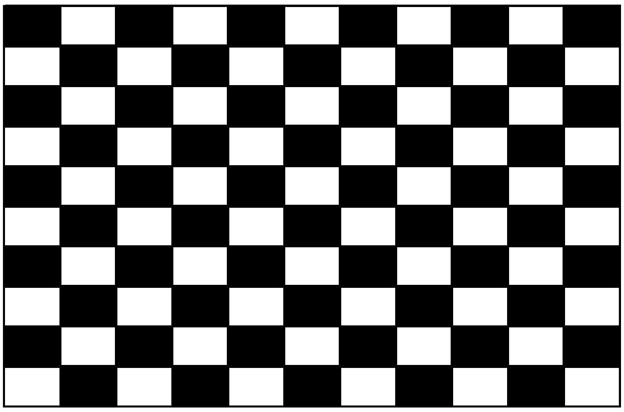
$\therefore$ Smallest number is 184

$= 1^2 + 8^2 + 4^2 = 81$

26. What is the number of ways in which one can choose 60 unit squares from a 11×11 chessboard such that no two chosen squares have a side in common?

Ans. (62)

Sol.



There are 121 unit squares of two colours 60 + 61.

if we want to choose square with no common sides then we have to select square of same colour.

Therefore if we choose white colour there is only one way to select 60 square from 60 squares and if we choose square of black colour then there are 61 ways to select 60 squares out of 61.

So total number of ways = 1 + 61 = 62

27. What is the number of ways in which one can colour the squares of a 4×4 chessboard with colous red and blue such that each row as well as each column has exactly two red squares and two blue squares?

Ans. (90)

Sol. Each row can be coloured in any one of the following six ways

RBRB, RRBB, RBBR, BBRR, BRRB and BRBR

First two rows can be coloured in $6 \times 6 = 36$ ways

Let us divide all 36 ways in three cases.

(i) First and second row are identical :

There are 6 such cases then last two rows can be painted in only 1 way.

Number of such such ways = 6

(ii) First and second row do not match at any palce :

Colour first row by any one of the 6 ways and switch colour in second row for corresponding squares.

First two can be coloured in 6 ways.

Now 3[rd] row can be painted in any one of the 6 ways and final row one way.

$\Rightarrow$ $6 \times 6 = 36$ ways in this case

(iii) First and second row match exactly at two places:

There are $36 - 6 - 6 = 24$ such cases

the column in which two square are of same colour in first two row can be painted in only one way (in third and fourth row) and the remaining two squares of third row can be painted in two ways last row will be fix

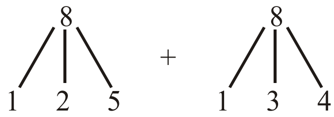
$\Rightarrow 24 \times 2 = 48$ ways

Hence total ways are $6 + 36 + 48 = 90$ ways.

28. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N.

Ans. (24)

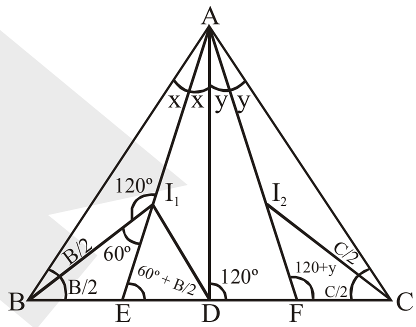
Sol. Total ways of distribution = $\left(\frac{8!}{1!2!5!}+\frac{8!}{1!3!4!}\right)\times 3!$ = 2688 ways



$\therefore$ sum of digits = 2 + 6 + 8 + 8 = 24

29. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60^\circ$, what is the measure of $\angle CI_2F$ in degrees?

Ans. (30)



Sol. $\frac{B}{2} + x = 60^\circ$ in ΔABI_1

$\therefore \angle AED = 60^\circ + \frac{B}{2}$ (exterior angle)

$\angle ADC = 60^\circ + \frac{B}{2} + x = 60^\circ + 60^\circ = 120^\circ$ (exterior angle)

In ΔADC, $2y + C = 60^\circ$

$\Rightarrow \ y + \frac{C}{2} = 30^\circ$

But In ΔCI_2F, $\angle I_2FC = 120^\circ + y$ (exterior angle)

In ΔCI_2F we have

$\angle I_2FC + \angle FCI_2$

$= 120^\circ + y + \frac{C}{2} = 120^\circ + 30^\circ = 150^\circ$ so $\angle CI_2F = 30^\circ$

30. Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial in which a_i is a non negative integer for each $i \in (0, 1, 2, 3, \ldots., n)$. If P(1) = 4 and P(5) = 136, what is the value of P(3)?

Ans. (34)

Sol. $p(1) = a_0 + a_1 + a_2 \ldots. a_n$

$4 = a_0 + a_1 + a_2 \ldots.. a_n$

$p(5) = a_0 + 5a_1 + 25a_2 + 125a_3 + 625a_4 + \ldots.. 5^na_n$

$136 = a_0 + 5a_1 + 25a_2 + 125a_3 + 625a_4 + \ldots.. 5^na_n$

observe that $a_4, a_5 \ldots a_n = 0$ as given that a_i is non negative integer.

$\therefore$ So we assume

$p(x) = x^3 + 2x + 1$

$p(1) = 4$

$p(5) = 125 + 10 + 1 = 136$

$p(3) = 27 + 6 + 1 = 34$