33^{rd} Indian National Mathematical Olympiad-2018

Date of Examination : 21^{th} January, 2018

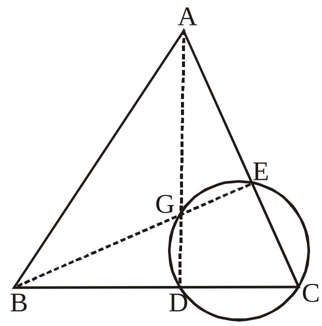
SOLUTIONS

1. Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points BC and CA ; let G be the centroid of triangle ABC. Suppose D, C, E, G are concyclic. Find the least possible perimeter of triangle ABC.

Sol. BD.BC = BG.BE

$$\frac{a}{2}.\ a = \frac{2}{3} m_b.m_b$$

$$\Rightarrow\ m_b^{\ 2} = \frac{3}{4}a^2 \qquad ...(1)$$



By appolonius theorem

$$a^2 + c^2 = 2\left(m_b^2 + \frac{b^2}{4}\right)$$

$$\Rightarrow\ m_b^2 = \frac{2a^2+2c^2-b^2}{4} \quad ...(2)$$

From (1) and (2)

$$\Rightarrow\ \frac{2a^2+2c^2-b^2}{4} = \frac{3}{4}a^2 \ \text{(From (1) and (2))}$$

$$\Rightarrow\ \boxed{2c^2 = a^2 + b^2} \qquad ...(3)$$

$\Rightarrow$ $a^2 + b^2$ must be even

$\Rightarrow$ a, b must be of same parity.

Now $c^2 = \dfrac{a^2+b^2}{2} = \left(\dfrac{a+b}{2}\right)^2 + \left(\dfrac{a-b}{2}\right)^2$

W.L.O.G let $a \ge b$

$\dfrac{a+b}{2} = x \in N$, $\dfrac{a-b}{2} = y \in N$

(as a, b of same parity)

$\Rightarrow$ $c^2 = x^2 + y^2$

For $y = 0$, $c = x$

$\Rightarrow a = b$ and $c = \dfrac{a+b}{2}$

$\Rightarrow c = a = b$ equilateral Δ which is not the case.

$\Rightarrow y > 0$

Now c, x, y are sides of a right angle triangle and smallest pythagorean triple is 5, 4, 3 ; second smallest 5, 12, 13

For 5, 4, 3 we have

$c = 5$, $\dfrac{a+b}{2} = 4$, $\dfrac{a-b}{2} = 3$

$\Rightarrow a = 7$, $b = 1$

Not possible

For 13, 12, 5 we have

$c = 13$, $\dfrac{a+b}{2} = 12$, $\dfrac{a-b}{2} = 5$

$c = 13$ and $a = 17$, $b = 7$

as $17 < 7 + 13$

least perimeter of ΔABC will be $7 + 13 + 17 = 37$

2. For any natural number n, consider a $1 \times n$ rectangular board made up of n unit squares. This is covered by three types of tiles 1×1 red tile, 1×1 green tile and 1×2 blue domino. (For example, we can have 5 types of tiling when $n = 2$: red-red; red-green; green-red; green-green and blue.) Let t_n denote the number of ways of covering $1 \times n$ rectangular board by these three types of tiles. Prove that t_n divides t_{2n+1}.

Sol. Let r_n, g_n, b_n respectively be the number of $1 \times n$ tiles that end with a red, green and blue tiles. Clearly, $t_n = r_n + g_n + b_n$. To get a $1 \times (n + 1)$ tile ending in a red tile, we can append a 1×1 red tile to any of the above three. Hence $r_{n+1} = r_n + g_n + b_n$. Similarly, $g_{n+1} = r_n + g_n + b_n$. To get b_{n+1}, we need to append a blue tile to a $1 \times (n - 1)$ tile. Thus $b_{n+1} = r_{n-1} + g_{n-1} + b_{n-1}$.

Thus

$t_{n+1} = r_{n+1} + g_{n+1} + b_{n+1}$

$= (r_n + g_n + b_n) + (r_n + g_n + b_n) + (r_{n-1} + g_{n-1} + b_{n-1})$

$= t_n + t_{n-1}$

Thus we have recurrence relation $t_{n+1} - 2t_n - t_{n-1} = 0$ whose characteristic equation is $\lambda^2 - 2y - 1 = 0$. Thus has characteristic roots $1 \pm \sqrt{2}$. Thus $t_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n = A\alpha^n + B\beta^n$, where $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. Since $t_1 = 2$ and $t_2 = 5$, we get $A = \dfrac{\alpha}{2\sqrt{2}}$ and $B = -\dfrac{\beta}{2\sqrt{2}}$. Thus

$$t_n = -\frac{\alpha^{n+1} - \beta^{n+1}}{2\sqrt{2}}$$

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Now,

$$t_{2n+1} = -\frac{\alpha^{2n+2} - \beta^{2n+2}}{2\sqrt{2}}$$

$$= \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha^{n+1} + \beta^{n+1})}{2\sqrt{2}}$$

$$= \left(\frac{\alpha^{n+1} - \beta^{n+1}}{2\sqrt{2}}\right)(\alpha^{n+1} + \beta^{n+1})$$

$$= t_n(\alpha^{n+1} + \beta^{n+1})$$

Note that

$$\alpha^{n+1} + \beta^{n+1} = \left(1+\sqrt{2}\right)^{n+1} + \left(1-\sqrt{2}\right)^{n+1} = 2\left(1 + \binom{n+1}{2}.2 + \binom{n+1}{4}.2^2 + ...\right)$$

is an integer and t_{2n+1} is divisible by t_n.

3. Let Γ_1 and Γ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ is an obtuse angle. Let the circumcircle of triangle O_1AO_2 intersect Γ_1 and Γ_2 respectively in points $C(\neq A)$ and $D(\neq A)$. Let the line CB intersect in Γ_2 in E; let the line DB intersect Γ_1 in F. Prove that the points C, D, E, F are concyclic.

Sol. **Claim :** CB passes through O_2 and DB through O_1

Proof : For circle Γ_1, $\angle AO_1O_2 = \dfrac{1}{2}\angle AO_1B = \angle ACB$ – (1)

Also for circle Γ_3 $\angle AO_1O_2 = \angle ACO_2$ (2)

From (1) and (2) we set

$\angle ACB = \angle ACO_2 \Rightarrow CB \parallel CO_2$

$\Rightarrow$ CB passes through O_2

Similarly BD, passes through O_1

Now $\angle BAE = 90°$ (as BF diameter of Γ_1)

and $\angle BAF = 90°$ (as BF diameter of Γ_1)

$\Rightarrow$ FAE are collinear and $\parallel$ to O_1O_2

Let $\angle FEC = \alpha \Rightarrow O_1O_2B = \alpha$ (as $O_1O_2 \parallel$ FE)

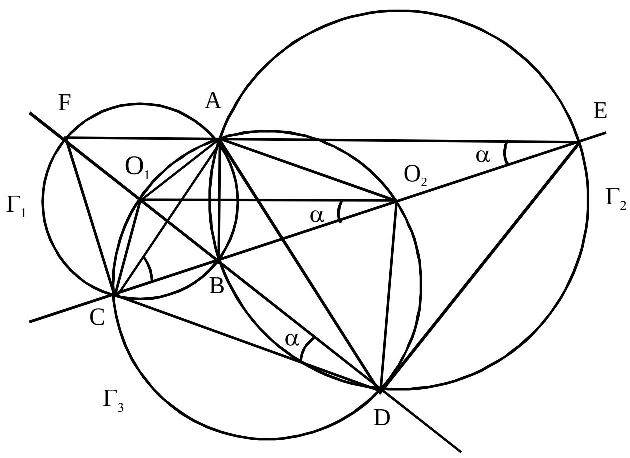
or $\angle O_1O_2C = \alpha$

$\Rightarrow$ $\angle O_1D_2C = \alpha$ (on Γ_3)

or $\angle FDC = \alpha$

$\Rightarrow$ $\angle FEC = \alpha = \angle FDC$

$\Rightarrow$ CDEF are concyclic.



4. Find all polynomials with real coefficients P(x) such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

Sol. Possibility (1) : P(x) is constant = c then

$P(x^3 - 1) = c$ and $P(x^2 + x + 1) = c$ and we are done.

Let P(x) be non contant polynomial.

As $P(x^2+x+1) | P(x^3-1)$

$\Rightarrow P(x^3 - 1) = P(x^2 + x + 1)\, Q(x)$ where

Q(x) in some polynomial in x.

$\Rightarrow P(x - 1)(x^2 + x + 1) = P(x^2 + x + 1)\, Q(x)$

$\Rightarrow$ Whenever $x^2 + x + 1$ in a root of P(x),

$(x - 1)(x^2 + x + 1)$ in also a root .... (1)

Let α be a root of P(x) such that $|\alpha|$ be maximum.

Now take $x^2 + x + 1 = \alpha \Rightarrow x = x_1x_2 =$ (say),

roots with $x_1 + x_2 = -1$,

$\Rightarrow$ Atleast one root out of x_1, x_2 will have distance more than 1 (from '1').

Let $|x_1-1| \le | \Rightarrow x_2 = -1-x_1$

$\Rightarrow |x_2-1| = |-1-x_1-1| = |3-(x_1-1)| \ge |3-|x_1-1|| \ge 2$

$= |x_2-1| > 1$ ....(2)

From one we have $(x_2 - 1)(x_2^2 + x_2 + 1) = (x_2 - 1)\alpha = \beta$ (say) is another root of P(x) = 0.

Here $|B| = |(x_2-1)\alpha| = |x_2-1||\alpha| \ge |\alpha|$

Which is a contradiction $\Rightarrow |\alpha| = 0 \Rightarrow \alpha = 0$

$\Rightarrow$ All root of non contant polonomial must be '0'.

$\Rightarrow$ $P(x) = a.x^n$, $a \in R$, $n \in N$.

An other solution $p(x) = c$, $c \in R$.

5. There are $n \ge 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbors combined, the teacher takes away one apple from that girl and gives one apple each to her neighbors. Prove that this process stops after a finite number of steps. (Assume that the teacher has an abundant supply of apples.)

Sol. Let a_i be number of apples held by girl i

i = 1, 2, ...n

$S_1 = a_1 + a_2 + \ldots\ldots\ldots + a_n$

$Q_1 = a_1^2 + a_2^2 + \ldots\ldots\ldots\ldots + a_n^2$

$a_k > a_{k+1} + a_{k-1}$ some $k \Rightarrow a_k - a_{k+1} - a_{k-1} \ge 1$

Then after 1^{st} step, $S_2 = S_1 + 1$ and $Q_2 \le Q_1 + 1$

As $(a_k - 1)^2 + (a_{k+1} + 1)^2 + (a_{k-1} + 1)^2 - a_k^{\ 2} + a^2_{\ k+1} + a^2_{k-1}$

$= 2(a_{k+1} + a_{k-1} - a_k) + 3 \le 1$

$\Rightarrow\ Q_2 - Q_1 \le 1 \qquad \Rightarrow\ Q_2 \le 1 + Q_1$

After r steps, $S_r = S_1 + r$ and $t_r \le t_1 + r$

Now apply power mean inequality after each step

$$\left(\frac{a_1^2 + a_2^2 + ... + a_n^2}{n}\right)^{1/2} \ge \frac{a_1 + a_2 + ... + a_n}{n}$$

then after r step we will be having :

$$\left(\frac{Q_1 + r}{n}\right)^{1/2} \ge \frac{S_1 + r}{n}$$

$\Rightarrow\ n(Q_1 + r) \ge (S_1 + r)^2$

$\Rightarrow\ r^2 - (2S_1 - n)r - n\ Q_1 \le 0$ ....(A)

as S_1, n, Q_1 is fix and r is variable we know that Eq. (A) can be true for finitly many r after that it will be false $\Rightarrow$ process stops after finite step.

6. Let N denote the set of all natural numbers and let $f : N \to N$ be a function such that

(a) $f(mn) = f(m)f(n)$ all m, n in N ;

(b) m + n divides f(m) + f(n) for all m, n in N.

Prove that there exists an odd natural number k such that $f(n) = n^k$ for all n in N.

Sol. $P(m, n) : f(mn) = f(m).f(n);\ Q(m, n) : m + n \mid (f(m) + f(n))$

$P(1, 1) : f(11) = f(1).\ f(1) \Rightarrow f(1) = 1$ as $f \in N$

$Q(2, 2) : 2 + 2 \mid (f(2) + f(2)) \Rightarrow\ 2 \mid f(2)$

$\Rightarrow f(2) = 2^k.q$, q some odd number $k \in N$.

If possible let q > 1 then there will exist a prime p such that p|q

$\Rightarrow$ p = odd prime.

Also we set p | f (2)

$$P\left(2.\frac{p-1}{2}\right) : f(p-1) = f\left(2.\frac{P-1}{2}\right) = f(2).f\left(\frac{p-1}{2}\right)$$

$\Rightarrow P \mid f(p - 1)$

Q (1, p – 1) : 1 + (p – 1) | (f(1) + f (p – 1)

$\Rightarrow p \mid (1 + f(p - 1) \Rightarrow p \mid 1$

Which is a contradiction $\Rightarrow q = 1 \Rightarrow f(2) = 2^k$

$\Rightarrow (2, 1) : (2 + 1) \mid (f(2) + f(1)) \Rightarrow 3 \mid (2^k + 1)$

$\Rightarrow 2^k + 1 \equiv 0 \pmod 3$

or $(-1)^k + 1 \equiv 0 \pmod 3$

$\Rightarrow$ k = odd.

Also from $f(mn) = f(m).f(n)$

$$\Rightarrow \quad \Rightarrow f\underbrace{(2.2.2.....2)}_{m\,times} = \underbrace{f(2).f(2)....f(2)}_{m\,times} = \underbrace{2^k.2^k...2^k}_{m\,times}$$

$\Rightarrow f(2^m) = (2^k)^m = 2^{km}$

Now $Q(n, 2^m)$: $(n + 2^m) \mid (f(n) + f(2^m))$

i.e. $(n + 2^m) \mid (f(n) + 2^{km})$ ....(1)

as $(x + y) \mid (x^k + y^k)$ for k = odd

$\Rightarrow n + 2^m \mid (n^k + (2^m)^k)$ ...(2)

From (1) and (2) we get

$(n + 2^m) \mid (f(n) + 2^{km}) - (n^k + 2^{km})) + m \in N$

$\Rightarrow (n + 2^m) \mid (f(n) - n^k) \ \forall \ m \in N$

$\Rightarrow f(n) - n^k$ has infinite divisors

$\Rightarrow f(n) - n^k = 0$

$\Rightarrow f(n) = n^k$ for some odd $k \in N$

Alternate solution

The answer is $f(x) = x^k$ for all $x \in N$, which k is an odd positive integer.

Throughout the solution, we'll donote $f(mn) = f(m) f(n)$ as (*, m, n)

and $m + n \mid f(m) + f(n)$ as (**, m, n).

(*, 1, 1) gives $f(1) = 1$

Claim 1 : $f(2) = 2^k$, and k is an odd positive integer.

Proof of clam 1 : Assume the contrary that for an odd prime p, $p|f(2)$. Then by *, we get $P \mid f(x)$ for all x even. Then since $p - 1$ is even we get $p \mid f(p - 1)$. Now (**, p – 1, 1) gives $p - 1 + 1 \mid f(p - 1) + f(1)$ $\Rightarrow p \mid f(1)$. Contradictory to $f(1) = 1$. This proves $f(2) = 2^k$.

Now (**, 2, 1) gives $2 + 1 \mid f(2) + 1 \Rightarrow 3 \mid 2k + 1 \Rightarrow$ k is odd. This proves Claim 1.

Claim 2 : $f(p) = p^m$ for any odd prime p.

Proof of claim 2 : Assume the contrary that for another prime $q \neq p$, $q \mid f(p)$. Then by *, we get $q \mid f(x)$ for all x which are multiplies of p.

We know there is a multiple of p which gives –1 as a residue when divided by q. Let that multiple of p be px, i.e., $px \equiv -1 \pmod q$.

Now (**, px, 1) gives $px + 1 \mid f(px) + f(1)$ and since we know $q \mid px + 1$ and $q \mid f(px)$, this gives $q \mid f(1)$. Contradicatory to $f(1) = 1$. This proves claim 2.

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Claim 3 : If $f(2) = 2^k$ and $f(p) = p^m$ for an odd prime p, then $k = m$

Proof of claim 3 : Assume the contrary that $k \neq m$. Then $c = |k - m| > 0$.

Take any positive integer a. By *, we know $f(2^a) = 2^{ak}$.

$(**, 2^a, p)$ gives $2^a + p \mid f(2^a) + f(p) \Rightarrow 2^a + p \mid 2^{ak} + p^m \Rightarrow 2^a + p \mid 2^{ak} + p^k - p^k + p^m$.

Since k is odd, we have $2^a + p \mid 2^{ak} + p^k$, which means $2^a + p \mid p^m - p^k \Rightarrow 2^a + p \mid p^c - 1$

since $\gcd(2, p) = 1$

Now notice that we got $2^a + p|p^c - 1$ for all positive integers a. But if we take 'a' large enough, clearly this will be false. Contradiction. Then $c = |k - m| = 0$. This proves claim 3.

So, we have proved $f(p) = p^k$ for all primes p, where k is an odd integer. By * this means $f(x) = x^k$ for all positive integers x, as desired. This completes the proof.
